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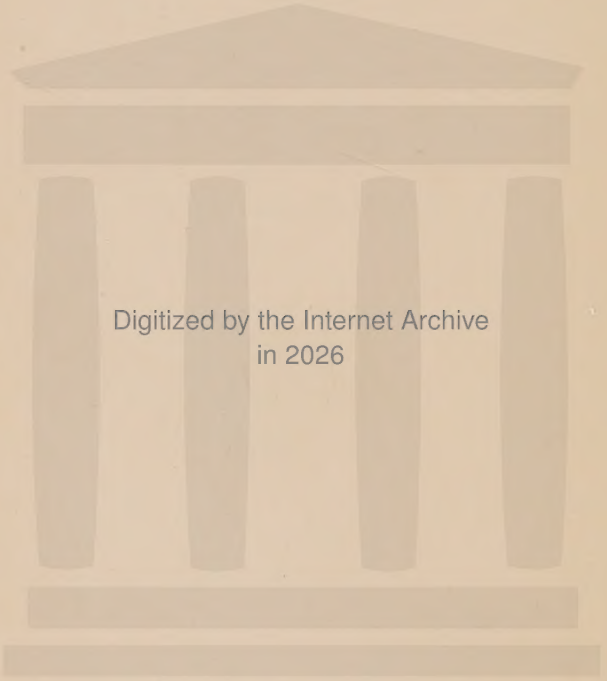

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# RECREATIONS IN MATHEMATICS

*By* H. E. LICKS

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# PREFACE

THE object of this book is to afford recreation for an idle hour and to excite the interest of young students in further mathematical inquiries. The topics discussed have therefore been selected with a view toward interesting students and mathematical amateurs, rather than experts and professors.

The Table of Contents is logically arranged with respect to chapters; but it will be found that within the latter, the topics are subject to no regular law or order. Some of these are long, others short; some are serious, others are frivolous; some are logical, others are absurd. It is feared that many things which might have been included have been omitted, and that still others which should have been omitted, have been included. The indulgence of readers is craved for this seeming lack of consistency and it is submitted in extenuation that the very character of the subject, partaking as it does somewhat of the nature of the curio collection, renders a more orderly treatment practically impossible.

The subject matter has been collected from many and divers sources and it is hoped that in spite of the complex nature of the work, the selection will appeal to the readers to whom it is addressed.

H. E. L.

*December, 1916.*

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# RECREATIONS IN MATHEMATICS

## CHAPTER I

### ARITHMETIC

### 1

OUNTING a series of things and keeping tally of the tens on the fingers were processes used by primitive peoples. From the ten fingers arose ultimately the decimal system of numeration. Recording the results of counting was done by the Egyptians and other ancient nations by means of strokes and hooks; for one thing a single stroke | was made, for two things two strokes || were used, and so on up to ten which was represented by ∩. Then eleven was written |∩, twelve ||∩, and so on up to twenty, or two tens, which was represented by ∩∩. In this way the numeration proceeded up to a hundred, for which another symbol was employed.

Names for ||, |||, ||||, ∩∩, etc., appear in the Egyptian hieroglyphics, but a special symbol for each name is not used. Probably the Hindoos first invented such symbols, and passed them on to the Arabs, through whom they were introduced into Europe.

### 2

### GREEK NOTATION

The Greeks used an awkward notation for recording the results of counting. The first nine letters of the Greek alphabet denoted the numbers from one to nine, so that $\alpha$

represented one, $\beta$ two, $\gamma$ three, and so on. Then the following nine letters were used for ten, twenty, thirty, etc., so that $\kappa$ represented ten, $\lambda$ twenty, $\mu$ thirty, and so on. Then the remaining letters $\tau$, $\upsilon$, etc., were used for one hundred, two hundred, etc., but as the Greek alphabet had only twenty-four letters, three symbols were borrowed from other alphabets. This was an awkward notation, and there seems to have been little use made of it except to record results. A number having two letters was hence between ten and one hundred, and one having three letters was between one hundred and one thousand; thus, $\lambda\delta$ was twenty-four and $\tau\kappa\delta$ was one hundred and fourteen. The Greeks were good mathematicians, as appears from their work in geometry, but only a few writers used this arithmetical numeration in computations, saying, for example, that the sum of $\kappa\alpha$ and $\lambda\beta$ was $\mu\gamma$. In those days the abacus or swan pan, similar to that seen in Chinese laundries in the United States, was employed to make arithmetical computations. From very early days this simple apparatus has been used throughout the East, and it is said that computations are made on it with great rapidity.

## 3

## ROMAN NUMERATION

The Romans represented the first five digits by I, II, III, IIII, and V, a V prefixed to the first four gave the digits from six to nine, while ten was represented by X, fifty by L and one hundred by C. This notation is still in use for a few minor purposes, it being modified by using IV for four, IX for nine, etc.; when a watch face is lettered in this notation, however, IIII is always used for IV, because Charles V said that he would allow nothing to precede a V. The

Roman notation was employed only to record numbers, and it does not appear that arithmetical operations were ever conducted with it. Perhaps this awkward notation retarded the development of mathematics among the Romans.

Frontinus, a Roman water commissioner, wrote in 97 A.D. a treatise on the Water Supply of the City of Rome, a translation of which, with an excellent commentary by Clemens Herschel, was published at Boston in 1889. A long list of the dimensions of the water pipes then in use is given, these being expressed in digits and fractions. The fraction 1/12 was denoted by a single horizontal stroke —, 2/12 by two strokes =, 3/12 by three strokes ≡, and so on up to 5/12. Then 1/2 was represented by *S*, while the fractions from 7/12 to 11/12 were represented by adding strokes to the *S*, thus, *S*=, *S*≡, indicated 1/2 + 4/12 or 5/6. The fraction 1/24 was indicated by Ɔ. The smallest fraction used was 1/288 which was represented by Ↄ. The following is the description of the pipe No. 50 given by Frontinus:

Fistula quinquagenaria: diametri digitos septem S≡—ƆↃ quinque, perimetri digitos XXV ƆↃ VII, capit quinarias XLS≡ƆↃV.

Of which Herschel's translation is as follows:

The 50-pipe: seven digits, plus 1/2, plus 5/12, plus 1/24, plus 5/288 in diameter; 25 digits, plus 1/24 plus 7/288 in circumference; 40 quinarias, plus 1/2, plus 2/12, plus 1/24, plus 5/288 in capacity.

The digit was one-sixteenth of a Roman foot and the quantity of water flowing through a pipe of $1\frac{1}{4}$ digits in diameter was called a quinaria. Frontinus takes the quantities of water flowing through pipes as proportional to the squares of their diameters, for he says that pipes of $2\frac{1}{2}$ and $3\frac{3}{4}$ digits in diameter discharge four and nine quinarias respectively.

## 4

## THE ARABIC SYSTEM OF NUMERATION

The Arabic method, by which the symbols, 1, 2, 3, etc., were used for the first nine integers, seems to have first originated in India, from whence it was carried by the Arabs to Europe, about the year 1200.

Long before this time Greek and Arabic astronomers had used the sexagesimal system in the division of the circle, and this, with Arabic numerals, was employed about 1200 in Europe for expressing numbers not at all connected with a circle. Thus, $28, 32' 17'' 45''' 20^{IV}$ meant 28 units plus 32/60, plus 17/3600, plus 45/216000, plus 20/1296000. This method of expressing fractions was certainly more convenient than the Roman method as used by the water commissioner Frontinus.

How the Arabic method of numeration was introduced into Europe is told by Ball in the following interesting account of one of the early Italian mathematicians.

## 5

## LEONARDO DE PISA

From Ball's Short Account of the History of Mathematics. Fourth Edition (London, 1908), pages 167–170.

Leonardo Fibonacci (i.e., filius Bonacci), generally known as Leonardo of Pisa, was born at Pisa about 1175. His father Bonacci was a merchant, and was sent by his fellow-townsmen to control the custom-house at Bugia in Barbary; there Leonardo was educated, and he thus became acquainted with the Arabic or decimal system of numeration, as also with Alkariami's work on Algebra. It would seem that Leonardo was entrusted with some duties, in connection

with the custom-house, which required him to travel. He returned to Italy about 1200, and in 1202 published a work called *Algebra et almuchabala* (the title being taken from Alkariami's work), but generally known as the Liber Abaci. He there explains the Arabic system of numeration, and remarks on its great advantages over the Roman system. He then gives an account of algebra, and points out the convenience of using geometry to get rigid demonstrations of algebraical formulas. He shows how to solve simple equations, solves a few quadratic equations, and states some methods for the solution of indeterminate equations; these rules are illustrated by problems on numbers. The algebra is rhetorical, but in one case letters are employed as algebraical symbols. This work had a wide circulation, and for at least two centuries remained a standard authority from which numerous writers drew their inspiration.

The Liber Abaci is especially interesting in the history of mathematics, since it practically introduced the use of Arabic numerals into Christian Europe. The language of Leonardo implies that they were previously unknown to his countrymen: he says that having had to spend some years in Barbary he there learnt the Arabic system, which he found much more convenient than that used in Europe; he therefore published it "in order that the Latin race might no longer be deficient in that knowledge." Now Leonardo had read very widely, and had travelled in Greece, Sicily, and Italy; there is therefore every presumption that the system was then not commonly employed in Europe.

The majority of mathematicians must have already known of the system from the works of Ben Ezra, Gerard, and John Hispalensis. But shortly after the appearance of Leonardo's book Alfonso of Castile (in 1252) published

some astronomical tables, founded on observations made in Arabia, which were computed by Arabs, and which, it is generally believed, were expressed in Arabic notation. Alfonso's tables had a wide circulation among men of science, and probably were largely instrumental in bringing these numerals into universal use among mathematicians. By the end of the thirteenth century it was generally assumed that all scientific men would be acquainted with the system; thus Roger Bacon writing in that century recommends algorism (that is, the arithmetic founded on the Arab notation) as a necessary study for theologians who ought, he says, "to abound in the power of numbering." We may then consider that by the year 1300, or at the latest 1350, these numerals were familiar both to mathematicians and to Italian merchants.

So great was Leonardo's reputation that the Emperor Fredrick II stopped at Pisa in 1225 to test Leonardo's skill, of which he had heard such marvellous accounts. The competitors were informed beforehand of the questions to be asked, some or all of which were composed by John of Palermo, who was one of Fredrick's suite. This is the first time that we meet with an instance of those challenges to solve particular problems which were so common in the sixteenth and seventeenth centuries. The first question propounded was to find a number of which the square, when either increased or decreased by five, would remain a square. Leonardo gave an answer, which is correct, namely 41/12. The other competitors failed to solve any of the problems. (See No. 33 for a problem in Algebra.)

## 6

## EARLY ARITHMETIC IN ENGLAND

The earliest book on arithmetic printed in England was "The Grounde of Artes, by M. Robert Recorde, Doctor of Physik." First issued in 1540 it was republished in numerous editions until 1699. The following extracts from the edition of 1573 give an idea of the method of instruction.

*Master.* — If numbering be so common that no man can doe anything alone, and much less talke or bargain with other, but still have to doe with numbre; this proveth not numbre to be contemptible and vile, but rather right excellent and of high reputation, sithe it is the grounde of all mens affaires, so that without it no tale can be told, no bargaining without it can dully be ended, or no business that man hath, justly completed. . . . Wherefore in all great workes are Clerkes so much desired? Wherefore are Auditors so richly feed? What causeth Geometricians so highly to be enhaunced? Because that by numbre suche things they find, which else would farre excell mans minde.

*Scholar.* — Verily, sir, if it be so that these men by numbring their cunning doe attaine, at whose great workes most men doe wonder, then I see well that I was much deceived, and numbring is a more cunning thing than I take it to be.

*Master.* — If numbre were so vyle a thing as you did esteem it, then need it not to be used so much in mens communication. Exclude numbre, and answer to this question: How many years old are you?

*Scholar.* — Mum.

*Master.* — How many daies in a week? How many weeks in a yeare? What landes hath your father? How many men doth he keep? How long is it sythe you came from him to me?

*Scholar.* — Mum.

*Master.* — So that if numbre wante, you answer all by Mummes.

The master then goes on to show how useful numbers are in "Musike, Physike, Law, Grammer" and such like, and then proceeds to teach him numeration, addition, subtraction, and so on. The master explains and illustrates the process and then tests the scholar by requiring him to

perform an example, the latter explaining as he goes on and asking questions on doubtful points. Thus, in addition, after having explained the process of carrying, the master gives the scholar the numbers 848 and 186 to be added.

*Scholar.* — I must set them so, that the two first figures stand one ouver another, and the other each ouer a fellow of the same place. And so likewise of other figures, setting always the greatest numbre highest, thus, as followeth:

848
186

Then I must add 6 to 8 which maketh 14, that is mixt numbre, therefore must I take the digit 4 and write it under the 6 and 8, keeping the article 1 in my mind, thus:

848
186
4

Next that, I doe come to the second figures, adding them up together, saying 8 and 4 make 12, to which I put the 1 reserved in my mind, and that makethe 13, of which numbre I write the digit 3 under 8 and 4, and keep the article 1 in my mind, thus:

848
186
34

Then come I to the third figures, saying 1 and 8 make 9, and the 1 in my mind maketh 10. Sir, shall I write the cypher under 1 and 8?

*Master.* — Yea.

*Scholar.* — Then of the 10 I write the cypher under 1 and 8 and keep the article in my mind.

*Master.* — What needeth that, seeing there followeth no more figures?

*Scholar.* — Sir, I had forgotten, but I will remember better hereafter. Then seeing that I am come to the last figures, I must write the cypher under them, and the article in a further place after the cypher, thus:

848
186
1034

*Master.* — So, now you see, that of 848 and 186 added together, there amounteth 1034.

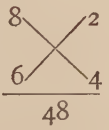
*Scholar.* — Now I think I am perfect in addition.

*Master.* — That I will prove by another example. There are two armies: in the one there are 106 800 and in the other 9400. How many are there in both armies, say you?

In those old days it seems that the multiplication table was learned only as far as five times five, and hence a process was necessary for multiplying together two numbers like 6 and 8. The following is the process as given by Recorde. The numbers 6 and 8 were placed on the left-hand side of a large letter X, thus:

Then each was subtracted from 10, the remainders placed directly opposite on the right-hand side, and a line was drawn under the whole, thus:

Next the units figure of the product was found by multiplying together the remainders 2 and 4, and the other figure of the product by subtracting crossways either 2 from 6 or 4 from 8; thus, 2 times 4 is 8, and 2 from 6 (or 4 from 8) is 4; therefore,




and hence six times eight makes forty-eight.

Napier's bones, used in England in the seventeenth century, consisted of nine sticks numbered at the top 1 to 9 inclusive, each stick having on its side the first nine multiples of the number at the top. These bones were hence

merely a multiplication table. When it was desired to multiply a number by 57, the sticks headed 5 and 7 were taken and their multiples used. Thus suppose that 89 was to be multiplied by 57. First, looking on the stick 5, the

$$\begin{array}{r} 89 \\ \underline{57} \\ 40 \\ 45 \\ 56 \\ \underline{63} \\ 5073 \end{array}$$

multiples of 5 by 8 and 9 were taken off and set down as shown, then looking on the stick 7 the multiples of 7 by 8 and 9 were taken off and set down; then the addition gave the product of 89 by 57. Thus were arithmetical operations performed in England less than four hundred years ago.

## 7

## THE SIGNS OF ARITHMETIC

The signs + and − are supposed to have been first used in Holland in the fifteenth century, to denote excess or deficiency in weight of bales of goods. The normal weight of a certain bale being, say, 4 centners, it was marked 4 c. + 5 lb. if it weighed 5 lb. more than the normal, and 4 c. − 5 lb. if it weighed 5 lb. less. These signs were used in a similar sense in Widman's Arithmetic published at Leipzig in 1489. It was not until about 1540 that they were used as signs of operation, that is, as directions to perform addition or subtraction.

The sign = was first used in works on Algebra, the earliest

mention being in Recorde's Whetstone of Wit issued in 1557, his selection of that sign being because "no 2 thyngs can be moare equalle."

The decimal point came much later, for fractional numbers were generally written in the duodecimal or sexagesimal form prior to the fifteenth century, as has already been explained. Napier and Briggs, the inventors of logarithms seem to have been the first to use, about 1620, the decimal method and the decimal point, although at first there was no point, but a line was drawn under all the decimals.

It is scarcely more than a hundred years since the decimal point came to be generally used in the United States. For example, Willett's Scholar's Arithmetic, used in the public schools of New York City was issued in a fourth stereotype edition in 1822. On page 23 it is said that in adding sums of money, the dollars, cents, and mills should be kept separate by placing a point between them, but the point used is a colon. On page 24 in subtraction, it is said that dots must be used to keep these units separate, but the dot used is a comma. Under multiplication the colon is used in some examples and the comma in others. Under division (page 27) the comma is used, and also numbers like $56.43 are written $56 43cts. Under "Reduction of Federal Money" on page 55, the single parenthesis is used as a decimal point; the problem being to reduce 387652 mills to dollars, the number is first divided by ten and the result stated as 38765(2; then this is divided by 100 giving 387(65:2, and finally the answer is given as 387 65cts 2m. Nothing more is said about decimals until we come to "Decimal Fractions" on page 151, and there the period is formally introduced as the decimal point, and the operations on numbers containing decimals are well explained;

even then it seems necessary to mention that a quantity like $590.217 means 590 dols. 21 cts. 7 m. A large part of the time of the children who used this book was devoted to intricate problems concerning pounds, shillings, pence, and farthings.

The signs $\times$ and $\div$ to indicate the operations of multiplication and division were not in common use before 1750. Prior to this date parentheses were not used in a case like $a(b + c + d)$, but a straight line was drawn over the $b + c + d$.

The use of the shilling mark / to indicate division is comparatively recent, it having been first employed about 1860. In this country it was rarely used until after 1890, but is now very commonly found in algebraic notation, and it will generally be used in the later chapters of this volume. Thus, 4/261 has the same meaning as $\frac{4}{261}$ or $4 \div 261$. This new division mark is of especial advantage in simplifying printed work, either in setting algebraic expressions in type or in writing fractions with a typewriter.

## ARITHMETIC AMUSEMENTS

### 8

Multiply 37037037 by 18; also multiply it by 27.
Multiply 1371742 by 9; also multiply it by 81.
Multiply 98765432 by 9; also multiply it by 1 1/8.

### 9

Think of any number, multiply it by 2, then add 4, multiply by 3, divide by six, subtract the number you thought of, and the result will be 2.

### 10

Think of any number, and add 1 to it, then square these two numbers and subtract the less from the greater. Now if you will tell me this difference, I can easily know the number you thought of, for I merely subtract 1 from the number you give me, then divide by 2, and the result is the number of which you thought.

### 11

To find the age of a man born in the nineteenth century. Ask him to take the tens digit of his birth year, multiply it by ten and add four; to this ask him to add the units figure of his birth year and tell you the result. Subtract this from 124 and you will have his age in 1920. Thus for a man born in 1848, $4 \times 10 = 40$, $40 + 4 = 44$, $44 + 8 = 52$, $124 - 52 = 72$, which will be his age in 1920, if then living.

### 12

Ask a person to multiply his age by 3, add 6 to the product, then divide the last number by three and tell you the result. Subtract two from that result and you have his age.

### 13

A woman goes to a well with two jars, one of which holds 3 pints and the other 5 pints. How can she bring back exactly 4 pints of water?

### 14

By the help of the following table the age of a person under 21 can be ascertained:

Ask the person to tell you in which column his or her age occurs. Then add together the numbers at the tops of those columns and the sum will be the age. Thus, if a person says that his age is found in columns B and E, then $2 + 16 = 18$ which is his age.

| A | B | C | D | E |
|---|---|---|---|---|
| 1 | 2 | 4 | 8 | 16 |
| 3 | 3 | 5 | 9 | 17 |
| 5 | 6 | 6 | 10 | 18 |
| 7 | 7 | 7 | 11 | 19 |
| 9 | 10 | 12 | 12 | 20 |
| 11 | 11 | 13 | 13 | |
| 13 | 14 | 14 | 14 | |
| 15 | 15 | 15 | 15 | |
| 17 | 18 | 20 | | |
| 19 | 19 | | | |

## 15

All integral numbers are either prime or composite. A prime number is one which has no integral divisors except itself and unity. There is no simple method of ascertaining what numbers are prime except that of the "sieve of Eratosthenes." By this method the odd integral numbers are written in ascending order, thus, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, etc., then every third number after three, every fifth number after 5, every seventh number after 7, and so on, are crossed off, and those remaining are primes. Thus from the above numbers, 9, 15, 21, and 27 are first crossed off, then 15 and 25, and then 21; the remaining numbers 3, 5, 7, 11, 13, 17, 19, 23, 29, together with 2, are the prime numbers less than thirty. This process becomes very laborious when the numbers run into millions.

## 16

A perfect number is one which is equal to the sum of its divisors. Thus, 6 equals $1 + 2 + 3$ and 1, 2, and 3 are the divisors of 6. The next perfect number is 28, the third one

is 496, and the fourth one is 8128. Beyond this there are no perfect numbers until 33 550 336 is reached. Then come 8 589 869 056, 137 438 691 328 and 2 305 843 008 139 952 128. This seems to be the largest perfect number thus far found. All of the above numbers end in 6 or 28. It is not known whether or not an odd number can be perfect, but the indications are against this being the case. The above numbers are taken from Ozanam's Recréations Mathématiques, published at Paris in 1750. Other numbers which he gives are not perfect, because he unfortunately made errors in computing them. Ozanam's book was first published in 1698; it passed through many editions and was also translated into English.

## 17

At rare intervals natural calculating boys come to public notice. One of these was Zerah Colburn who was born in New England in 1804 and taken to London when eight years old to exhibit his powers. He could mentally multiply any number less than 10 into itself successively nine times and give the results faster than they could be written down. He was asked what number multiplied by itself gave 106 989 and he instantly replied 327. With equal promptness he stated that the number which multiplied twice by itself gave 268 336 125 was 645, this being a problem in cube root for which an ordinary computer would require several minutes. He was asked to name a number which would divide exactly 36 083 and he immediately replied that there was no such number, in other words he recognized this as a prime number just as readily as we recognize 29 or 37 to be one. He could very quickly multiply together two numbers of four or five figures, and perform

many other remarkable mental feats. These natural calculators are rarely able to explain their processes, and their powers fade away and disappear as they grow up and become educated.

18

How many people know that the square of 3 plus the square of 4 equals the square of 5? All surveyors and draftsmen know it, also most machinists and carpenters, but to those in other trades it is probably quite unknown.

Another interesting arithmetical theorem is that the cube of 3 plus the cube of 4 plus the cube of 5 equals the cube of 6. Probably few students who read this book have ever before heard of this important relation.

19

In one hand a person has an odd number of coins or pebbles and in the other hand an even number, the knowledge of the same being unknown to anyone except himself. Ask him to multiply the number in the right hand by 2 and the number in the left hand by 3. Then ask him to add together the two products and tell you their sum. If this sum is odd the left hand has the odd number of coins, but if the sum is even, the left hand has the even number of coins.

20

One tumbler is half full of wine and another tumbler is half full of water. A teaspoonful of wine is taken from the first tumbler and put into the other one. Then a teaspoonful of the mixture is taken from the second tumbler and put into the first one. Is the quantity of wine removed from the first tumbler greater or less than the quantity of water removed from the second tumbler? Ball in his Mathe-

matical Recreations and Essays says that the majority of people will say it is greater, but that this is not the case. H. E. Licks, who has studied this problem, claims that the two quantities are exactly equal.

### 21

A stranger called at a shoe store and bought a pair of boots costing six dollars, in payment for which he tendered a twenty-dollar bill. The shoemaker could not change the note and accordingly sent his boy across the street to a tailor's shop and procured small bills for it, from which he gave the customer his change of fourteen dollars. The stranger then disappeared, when it was discovered that the twenty-dollar note was counterfeit, and of course the shoemaker had to make it good to the tailor. Now the question is, how much did the shoemaker lose?

### 22

At an humble inn where there were only six rooms, seven travellers applied for lodging, each insisting on having a room to himself. The landlord put the first man in room No. 1 and asked one of the other men to stay there also for a few minutes. He then put the third man in room number two, the fourth man in room No. 3, the fifth man in room No. 4, and the sixth man in room No. 5. Then returning to room No. 1 he took the seventh man and put him in room No. 6. Thus each man had his own room!

### 23

An Arab merchant directed by will that his seventeen horses should be divided among his three sons, one-half of them to the eldest, one-third to the second son and one-ninth to the youngest son. How to make the division was

a serious problem, for the eldest son claimed nine horses, but the others objected because this was more than one-half of seventeen. In this dilemma they applied to the Sheik who put his white Arabian steed among the seventeen horses, directed the eldest son to take one-half of the eighteen or nine, the second son to take one-third of the eighteen or six, and the youngest son to take one-ninth of the eighteen or two. Thus, since nine plus six plus two are seventeen, the horses were divided satisfactorily among the three sons. "Now," said the Sheik, "will I take away my own horse," and he led the Arabian steed back to his peg in the pasture.

**24**

To add 5 to 6 in such a way that the sum may be 9. Make six marks at equal distances apart, thus //////. Between the first and second marks draw a slanting line so as to form the letter *N*; then do the same between the fourth and fifth marks; finally add to the last line three horizontal marks so as to form the letter E. Then the problem is solved, for the five marks added to the given six marks have made *NINE*.

Another interesting problem in this line is to add three marks to a given five so as to make a quotation from Shakespeare. The added three marks give *KINI*, and you ask, where in this is found the required quotation. After a few minutes silence I reply, "A little more than kin but less than kind."

**25**

Two impossible problems: (1) If 3 is one-third of 10, what is one-quarter of twenty? (2) A man who had a bale of cotton sold it for $50, bought it back for $45, and then sold it again for $65. What was the net gain to the man?

## 26

The Indian mathematician Sessa, the inventor of the game of chess, was ordered by the king of Persia to ask as a recompense whatever he might wish. Sessa modestly requested to be given one grain of wheat for the first square of the board, two for the second, four for the third, and so on, doubling each time up to the sixty-fourth square. The wise men of the king added the numbers 1, 2, 4, 8, 16, etc., and found the sum of the series to sixty-four terms to be 18 446 744 073 709 551 615 grains of wheat. Taking 9000 grains in a pint we find the whole number of bushels to be over 32 000 000 000 000, which is several times the annual wheat production of the whole world.

## 27

H. E. Licks once had a class of students well versed in arithmetic, algebra, trigonometry, and calculus, but not one of them could solve the following simple problem, as they knew nothing about bookkeeping. The problem is hence here given for other young people.

A Coal Company appointed an agent, agreeing to pay him a salary of $265 for six months, all of the coal at the end of that time and all of the profits to belong to the Company. The Company furnished him with coal to the amount of $825.60 and in cash $215.00. The agent received for coal sold $1323.40, paid for coal bought $937.00, paid sundry expenses authorized by the Company $129.00, paid his own salary $265.00, paid to the Company $200.00, delivered to indigent persons by order of the Company coal to the amount of $13.50. At the end of the six months the Company took possession and found coal amounting to

$616.50. The agent then paid to the Company the money belonging to them. How much did he pay? Did the Company gain or lose by the agency and how much?

## 28

### THE FIFTEEN PUZZLE

About the year 1880 everyone in Europe and America was engaged in the solution of this interesting puzzle. A square shallow box contained fifteen blocks numbered 1 to 15 inclusive and these could be moved about one block at a

Fig. 1.

Fig. 2.

Fig. 3.

time, on account of the blank space. The blocks being placed in the box at random, say as shown in Fig. 1, the problem was to arrange them in regular order in the manner shown in Fig. 2. It was a fascinating exercise to shift these blocks until 1 was brought to the upper left-hand corner, then to bring 2 next to it, and thus keep on until the regular order of Fig. 2 was secured. But sometimes it happened, when the lowest row was reached, that the order of Fig. 3 resulted; for this case mathematicians proved that it was impossible to cause the blocks to take the regular order of Fig. 2. Mathematical analysis also showed that for many random positions of the blocks (Fig. 1), one-half of them would result in the order of Fig. 2 and one-half in the order of Fig. 3.

There is, however, a way by which the arrangement of Fig. 3 can be brought into regular order. Move the blocks until the upper left-hand corner is blank and the blocks 1, 2, 3 fill the other spaces of the upper row, then continue until the blocks 4, 5, 6, 7 fill the second row and 8, 9, 10, 11 the third row, then the lowest row can be arranged in the order 12, 13, 14, 15. This is a solution of the puzzle, if the statement of the problem is merely that "the blocks are to be arranged in regular order."

This puzzle comes under that branch of arithmetic known as permutations and combinations, and much mathematical thought has been expended upon it. The number of ways in which the fifteen blocks can be put at random in the box (Fig. 1) is 1 307 674 368 000. In the early stages of the craze it was not recognized that half of these combinations lead to the result of Fig. 2 and half of them to the result of Fig. 3. Hence when Fig. 3 was reached, the player usually kept on trying to obtain the arrangement of Fig. 2. Finally, after mathematicians had proved that it is impossible to bring Fig. 3 to agree with Fig. 2, the craze abated.

It has been stated that this interesting puzzle was invented in 1878 by a deaf and dumb man as a solitaire game. In the height of the craze persons in public conveyances could be seen every day attempting to solve the puzzle. Some physicians thought that this work was a beneficial mental exercise, but others claimed that it led to nervous disorders. A poet well expressed the latter opinion as follows:

> Put away his crack-brain puzzle,
> He has climbed the asylum stair;
> Numbers thirteen, fifteen, fourteen,
> Turned his head and sent him there!

## CHAPTER II

## ALGEBRA

### 29

EUCLID used algebra in a geometric form, expressing the equations always by words. For example: if a straight line be divided into two parts, the square on the line is equal to the sum of the squares on the two segments plus twice the rectangle of those segments; this in modern algebra is the theorem $(a + b)^2 = a^2 + b^2 + 2\,ab$. Until the sixteenth century all algebraic equations were generally expressed in words; for instance, Omar Kayyam wrote about 1100, "Cubus, latera et numerus aequalis sunt quadratis," meaning $x^3 + bx + c = ax^2$. Cardan about 1550 wrote "Cubus $\bar{p}$ 6 rebus aequalis 20," meaning $x^3 + 6\,x = 20$. Ramus about the same time wrote $7\,q + 3\,l - 2$ where $l$ meant the unknown quantity and $q$ its square. At this date $R_3\,17$ was introduced which later became $\sqrt[3]{17}$; here the $R$ signified radix or root.

Vieta, the founder of modern algebra, wrote about 1580 $1\,C - 8\,Q + 16\,N$ aequ 40, signifying $x^3 - 8\,x^2 + 16\,x = 40$, since $C$ meant cube, $Q$ meant square and $N$ meant first power of the unknown quantity. Fifty years later Descartes introduced $x$ to represent the unknown quantity in the equation. Algebra as we now know it is scarcely more than three hundred years old.

30

The earliest note of an equation is found in the Egyptian records of Ahmes in the following form "heap, its two-thirds, its half, its seventh, its whole gives 97," that is, heap is the unknown quantity $x$, and $2/3\ x + 1/2\ x + 1/7\ x + x = 97$ is the equation to be solved. It was a long time, however, before equations of the second degree appeared, except in the geometric form of Euclid. Here the Greeks led the way, and the equation of the third degree appeared first in the famous problem of the duplication of the cube. This leads to $x^3 = 2\ a^3$ where $x$ is the edge of a cube having a volume double that of the cube whose edge is $a$.

31

Algebra is a fascinating study with a notation and rules all its own. It is easier than geometry, because reasoning on the original problem is not required at every step, this reasoning being done automatically, as it were, by the operations on the symbols. It is a kind of generalized arithmetic whose rules and operations throw much light on the process of common arithmetic. The first important points to be learned are the properties of the signs + and − in multiplication, and when the student thoroughly understands that $-a$ multiplied by $-b$ gives $+ab$, he has entered upon a new field of intellectual pleasure. All young people ought to know something of the elements of algebra.

## ALGEBRAIC AMUSEMENTS

32

Methods much in vogue a hundred years ago for solving simple equations were those called Single Position and

Double Position. The former may be illustrated by the following problem: What number is it of which its double, its half, and its third are equal to 34? Assume the number to be 48; then its double is 96, its half is 24, and its third is 16, and the sum of these is 136. Then state the proportion, the required number is to the assumed number as the given result is to the computed result, or $x : 48 :: 34 : 136$ from which the required number is 12. Why this proportion gives the correct result was rarely explained, so that the method was one of memory rather than of reasoning.

33

One of the problems put to Leonardo de Pisa at the contest before the Emperor Fredrick II in 1225 (No. 5) was as follows, the letters $u$, $x$, $y$, $z$, being introduced to simplify the enunciation: Three men $A$, $B$, $C$, possess a sum of money $u$, their shares being in the ratio $3 : 2 : 1$. $A$ takes away $x$, keeps half of it, and deposits the remainder with $D$; $B$ takes away $y$, keeps two-thirds of it, and deposits the remainder with $D$; $C$ takes away all that is left, namely $z$, keeps five-sixths of it, and deposits the remainder with $D$. The deposits with $D$ are found to belong to $A$, $B$, $C$, in the proportions $3 : 2 : 1$. Find $u$, $x$, $y$, $z$. Leonardo showed that the problem has many solutions, one of these being $u = 47$, $x = 33$, $y = 13$, $z = 1$.

34

A positive quantity $a^2$ has two square roots $+a$ and $-a$. But a negative quantity $-a^2$ has no square roots except the imaginary ones $+ a\sqrt{-1}$ and $- a\sqrt{-1}$. The imaginary quantity $\sqrt{-1}$ first came to light through algebraic equations. Thus the equation $x^2 - 6x + 11 = 0$ has the two

roots $x = 3 + \sqrt{-2}$ and $x = 3 - \sqrt{-2}$, each of which satisfies the equation. Such roots were at first called impossible values, but later it was found that they have a good geometric representation, which will be spoken of later.

### 35

A quantity has three cube roots. For example, the three cube roots of 8 are $+2$, $-1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$, as may be easily verified. Likewise a quantity has four fourth roots, five fifth roots, and so on.

### 36

The symbol 0/0 indicates an indeterminate quantity whose value can be found by consideration of the quantities which give rise to it. What is the value of $(a^3 - b^3)/(a - b)$ when $a$ and $b$ are each equal to unity? Most beginners in algebra will reply 0. But, performing the division indicated, it will be found that $(a^3 - b^3)/(a - b)$ has the value $a^2 + ab + b^2$ which becomes 3 when $a = 1$ and $b = 1$. Hence for this case the value of 0/0 is 3.

### 37

The equation $a^x + b^x = c^x$, in which $a$, $b$, $c$ are integers, cannot be satisfied except when $x = 2$. For this value of $x$ there are numerous solutions, as may be seen in the next chapter. Fermat's "last theorem" states that integral values for $a$, $b$, $c$ cannot be found when $x$ is greater than 2. The equation $a^x + b^x + c^x = d^x$ can, however, be satisfied in integers when $x = 3$, and $3^3 + 4^3 + 5^3 = 6^3$ is one solution.

38

Ask a person to think of a number while you call it $x$, then ask him to multiply it by 2 and add 6 to the product; this gives you $2x + 6$. Then ask him to square the number he thought of and add it to this, which gives you $x^2 + 2x + 6$. Next he is to subtract 5 from this, which gives you $x^2 + 2x + 1$. He is now asked to take the square root of this, and you have $x + 1$. If now he gives you this result, you have only to subtract 1 from it to know the number he thought of at first. Endless exercises like this can be made by the use of a little algebra.

39

There is a rule for determining whether or not a given number is a prime, but it gives rise to such large numbers that its use is generally impracticable. Let $n!$ denote the product of the first $n$ integral numbers or $n! = 1 \times 2 \times 3 \times \ldots n$. Then if $n! + 1$ is exactly divisible by $n + 1$ the number $n + 1$ is a prime. For example, to find if 7 is a prime, let $n + 1 = 7$ or $n = 6$, then $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$, and $720 + 1 = 721$ which is exactly divisible by 7; hence 7 is a prime.

40

The following interesting problem is said to be due to Sir Isaac Newton: Three cows eat in two weeks all the grass on two acres of land, together with all the grass which grows there in the two weeks. Two cows eat in four weeks all the grass on two acres of land, together with all the grass which grows there in the four weeks. How many cows, then, will eat in six weeks all the grass on six acres of land together with all the grass which grows there in the six weeks? In

this problem it is, of course, understood that the quantity of grass on each acre is the same when the cows begin to graze, and also that the rate of growth is uniform during the time of grazing. Let $x$ be the quantity of grass which grows on one acre in one week. Then from the statements of the problem it is not difficult to show that $x$ is the same as the quantity of grass which was on one-fourth of an acre when the grazing began. The reader can now easily complete the solution without algebra and show that the answer to the problem is five cows.

## ALGEBRAIC FALLACIES

### 41

A fallacy is improper reasoning which leads to an absurd result. Some of the fallacies are very puzzling to a beginner. The improper use of the quantity $0/0$ lies at the basis of an extensive series of fallacies. For example, take the equation $15x + 12 = 6x + 30$. This may be written $15x - 30 = 6x - 12$ or $5(3x - 6) = 2(3x - 6)$; dividing by $3x - 6$ gives the absurd result $5 = 2$. Here the true solution is $3x - 6 = 0$ or $x = 2$. Hence the last equation is really $5 \times 0 = 2 \times 0$ or $5 = 2 \times 0/0$. The value of $0/0$ is not 1 as the previous division has incorrectly assumed, but it is $5/2$, so that the true result of the division is the correct conclusion $5 = 5$. The following fallacies rest upon this improper use of $0/0$ and it is left to the reader to detect the place where the incorrect reasoning begins.

### 42

To prove that the numbers 1 and 3 are equal. Let $a = b$, then $ab^2 = a^3$; subtracting $b^3$ from both members gives $ab^2 - b^3 = a^3 - b^3$; dividing by $a - b$ gives $b^2 = a^2 +$

$ab + b^2$; in this make $a = 1$ and $b = 1$, and there results $1 = 3$.

### 43

Any number $a$ is equal to a smaller number $b$. Let $c$ be their difference so that $a = b + c$; if this equation be multiplied by $a - b$, there is found

$$a^2 - ab = ab - b^2 + ac - bc$$

or

$$a^2 - ab - ac = ab - b^2 - bc.$$

Dividing both members of the last equation by $a - b - c$, there results $a = b$, which proves the proposition enunciated.

### 44

To prove that $-1 = +1$. Let $x^2 - 1 = 0$; divide by $x + 1$ and there is found $x - 1 = 0$ or $x = +1$; divide by $x - 1$ and then $x + 1 = 0$ or $x = -1$. Therefore $-1 = +1$.

### 45

To solve the equation $5 + \frac{9x - 55}{7 - x} = \frac{4x - 20}{15 - x}$. Reducing the first member so that $7 - x$ shall be the common denominator of both terms gives

$$\frac{4x - 20}{7 - x} = \frac{4x - 20}{15 - x} \quad \text{or} \quad \frac{1}{7 - x} = \frac{1}{15 - x},$$

from which results the theorem that $7 = 15$!

### 46

Another class of fallacies embraces those which neglect to consider that a quantity has two square roots of equal value except that one root is positive and the other nega-

tive. As an example take the true equation $16 - 48 = 64 - 96$; add 36 to each member giving $16 - 48 + 36 = 64 - 96 + 36$. Each member is now a perfect square or $(4 - 6)^2 = (8 - 6)^2$. Taking the square root of each side, gives $4 - 6 = 8 - 6$ or the absurd result $4 = 8$. The fallacy here lies in taking the wrong square root, the correct extraction for this case being $(4 - 6) = -(8 - 6)$ which gives the correct conclusion $-2 = -2$. The following fallacies are based upon the neglect to consider all the roots of a quantity.

Solve the equation $x + 2\sqrt{x} = 3$. Proceeding in the usual manner there are found $x = 1$ and $x = 9$. The first satisfies the equation, but the second does not. Will the reader explain?

## 47

To solve the equation $x - a = \sqrt{x^2 + a^2}$; squaring both sides and reducing gives $-2ax = 0$, whence $x = 0$. But this root does not satisfy the given equation and hence the solution cannot be correct. Where lies the fallacy?

Any two numbers are equal to each other. Let $a$ and $b$ be the two numbers and assume the equation $(x - a)^3 = (x - b)^3$. Taking the cube root of both members gives $x - a = x - b$, whence $a = b$. Perhaps the advanced student may be able to show that the equation $(x - a)^3 = (x - b)^3$ has the three roots $x = \frac{1}{2}(a + b) \pm \frac{1}{6}\sqrt{-3}(a - b)$ and $x = \frac{1}{2}(a + b)$.

To solve the equation $x - a = \left(x^2 - a\sqrt{x^2 + a^2}\right)^{\frac{1}{2}}$. Square each member twice and there will be found $x = 4/3\, a$ and $x = 0$. The first root satisfies the equation, but the second does not. Why not? Here it may be shown that $x = \infty$ is the true value of the second root.

48

Absurd fallacies like the following make good amusement for evening parties of young people: (1) Given the equations 32 ounces = 2 pounds, and 8 ounces = 1/2 pound; multiplying these equations, member by member, gives 256 ounces = 1 pound; where is the fallacy? (2) Each of the following statements is true; 1 cat has 4 legs, 0 cat has 2 legs; add these, member by member, and there is found 1 cat has 6 legs; where lies the fallacy?

Sometimes beginners make mistakes like the following: $\sqrt{2} + \sqrt{3} = \sqrt{6}$; $a^2 + a^3 = a^6$; $x^0 = 0$; $x^{12} \div x^2 = x^6$; $x^{\frac{3}{2}} + x^{\frac{2}{3}} = x$. Where is the error in each of these?

49

What is the sum of the infinite series $1 - 1 + 1 - 1 + 1 -$, etc.? If 1 be divided by $1 + x$ there is found:

$$1/(1 + x) = 1 - x + x^2 - x^3 + x^4 - x^5 +, \text{etc.},$$

and, making $x = 1$, it appears that the sum of the given infinite series is 1/2. But suppose we divide 1 by $1 + x + x^2$, giving

$$1/(1 + x + x^2) = 1 - x + x^3 - x^4 + x^6 - x^7 +, \text{etc.},$$

and then make $x = 1$, which shows that 1/3 is the sum of the given infinite series. Which value is correct, 1/2 or 1/3?

50

The theory of logarithms belongs in Algebra. Since $(-a)^2 = a^2$ it may be thought that $\log (-a)^2 = \log a^2$ or $2 \log (-a) = 2 \log a$, from which it appears that $\log (-1) = \log 1$, but this is not true on account of a concealed fallacy. Other interesting problems may also be stated:

(1) What is the value of $x$ in the equation $(3/4)^{\log x} + (4/3)^{\log x} = 25/12$? (2) What is the value of $x$ in $e^{ix} = 1$, where $e$ is the number 2.7128 and $i$ is $\sqrt{-1}$?

## 51

A woman came to town with a basket of eggs. To the first customer she sold half her eggs and half an egg. To the second customer she sold half of the remaining eggs and one-half an egg. To a third customer she sold half of the remaining eggs and one-half an egg. Then counting the eggs in her basket she found exactly three dozen. How many eggs had she at the start?

At first thought this problem is impossible, for the idea of a woman selling half an egg and then walking along with the other half seems absurd. But trying algebra, let $x$ equal the original number, then at the first transaction she sold $\frac{1}{2}x + \frac{1}{2}$ and had left $\frac{1}{2}x - \frac{1}{2}$. Thus proceeding to the last sale we put the final expression equal to 36 and find $x = 295$. One-half of 295 is $147\frac{1}{2}$ and $\frac{1}{2}$ an egg added makes 148. Thus she sold the first customer 148 eggs and had 147 left; to the second she sold $73\frac{1}{2} + \frac{1}{2}$ or 74 and then had 73 left; to the third she sold $36\frac{1}{2} + \frac{1}{2}$ or 37 and had 36 eggs left. Hence no division of an egg was necessary in making the three sales.

## 52

The following are problems proposed by the ancient Hindu mathematicians:

(1) The square root of half the number of bees in a swarm has flown out upon a jessamine bush, 8/9 of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus-flower into which he was

allured in the night by its sweet odor, but is now imprisoned in it. Tell me the number of the bees.

(2) A sixteen-year old girl slave costs 32 Uishkas, what costs one twenty years old by inverse proportion, the value of living creatures being regulated by their age, the older being the cheaper.

(3) Beautiful maiden with beaming eyes, tell me, as thou understandest the right method of inversion, what is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by 1/3 of the quotient, multiplied by itself, diminished by 52, by extraction of square root, addition of 8, and division of 10 gives the number 2?

(4) Of a flock of ruddy geese, ten times the square root of the number departed from the Manasa lake on the appearance of a cloud, an eighth part went to the forest of Sthalapadminis, and three couples were engaged in sport on the waters abounding with delicate flowers of the lotus. Tell me quickly, dear girl, the number of the flock.

## 53

It is said that the age of Diophantus when he died is known from the following problem: Diophantus was a child for one-sixth of his life, a youth for one-twelfth, and a bachelor for one-seventh; five years after his marriage a son was born who lived one-half as long as his father and who died four years before his father. When and where this problem was first propounded, I know not, but Cajori in his History of Elementary Mathematics says that it was an epitaph. Compute, young man, the age of Diophantus.

Plutarch relates the legend that the poet Homer died of vexation at being unable to solve a riddle propounded to

him by some young fishermen, in answer to his question as to how many they had caught. "As many as we caught," they said, "we left, as many as we did not catch, we carry."

## 54

## THE CATTLE PROBLEM OF ARCHIMEDES

Condensed from *Popular Science Monthly*, November, 1905.

Lessing, librarian at Wolfenbüttel, discovered there about 1770 the manuscript of a Greek poem which enunciated a problem of great difficulty. The name of Archimedes appears in the title of the poem, it being said that he sent the problem in a letter to Eratosthenes to be investigated by the mathematicians of Alexandria. It may well be doubted, however, if Archimedes was the real author, since no mention of the problem has been found in ancient writings.

The following statement of the problem has been abridged from the German translations published by Nesselman in 1842 and by Krumbiegel in 1880:

Compute, O friend, the number of the cattle of the sun which once grazed upon the plains of Sicily, divided according to color into four herds, one milk white, one black, one dappled, and one yellow. The number of the bulls is greater than the number of the cows and the relations between them are as follows:

White bulls = $(1/2 + 1/3)$ black bulls + yellow bulls.
Black bulls = $(1/4 + 1/5)$ dappled bulls + yellow bulls.
Dappled bulls = $(1/6 + 1/7)$ white bulls + yellow bulls.
White cows = $(1/3 + 1/4)$ black herd.
Black cows = $(1/4 + 1/5)$ dappled herd.
Dappled cows = $(1/5 + 1/6)$ yellow herd.
Yellow cows = $(1/6 + 1/7)$ white herd.

If thou canst give, O friend, the number of each kind of bulls and cows, thou art no novice in numbers, yet cannot be regarded as of high skill. Consider, however, the following additional relations between the bulls of the sun:

White bulls + black bulls = a square number.
Dappled bulls + yellow bulls = a triangular number.

If thou hast computed these also, O friend, and found the total number of cattle, then exult as a conqueror, for thou hast proved thyself most skilled in numbers.

It is seen that the exercise as stated involves two problems, the first to find integral numbers that satisfy the first seven conditions, and the second to find integral numbers that satisfy all nine conditions. The Wolfenbüttel manuscript has an Appendix giving 4 031 126 560 as the total number of cattle, but this satisfies only the first seven conditions. For this first problem let $W$, $B$, $D$, $Y$, represent the number of white, black, dappled, and yellow bulls, and let $w$, $b$, $d$, $y$, represent the number of white, black, dappled, and yellow cows. Then follow the seven equations:

$$W = \tfrac{5}{6} B + Y \quad (1) \qquad b = \tfrac{9}{20} (D + d) \quad (5)$$
$$B = \tfrac{9}{20} D + Y \quad (2) \qquad d = \tfrac{11}{30} (Y + y) \quad (6)$$
$$D = \tfrac{13}{42} W + Y \quad (3) \qquad y = \tfrac{13}{42} (W + w) \quad (7)$$
$$w = \tfrac{7}{12} (B + b) \quad (4)$$

and these contain eight unknown quantities. The problem, hence, is of the kind called indeterminate, for many sets of numbers may be found to satisfy the seven equations. That set having the smallest numbers is the one required, for any other set may be found by multiplying these by the same integer. If $B$ and $W$ are eliminated from equations (1), (2), (3) there will be found $891\ D = 1580\ Y$, and hence $D = 1580$ and $Y = 891$ are the smallest integral numbers satisfying it; from these are found $B = 1602$ and $W = 2226$. These numbers are now multiplied by an integer $m$ and substituted in equations (4) to (7); then proceeding with the elimination, it is found that 4657 is the least value

of $m$ that will make the results integral numbers. Accordingly,

$$\begin{aligned} B &= 7\,460\,514 & b &= 4\,893\,246 \\ W &= 10\,366\,482 & w &= 7\,206\,360 \\ D &= 7\,358\,060 & d &= 3\,515\,820 \\ Y &= 4\,149\,387 & y &= 5\,439\,213 \end{aligned}$$

are the least numbers satisfying the conditions of the first problem. The total number of cattle is 50 389 082, not too many to graze upon the island of Sicily, the area of which is about 7 000 000 acres.

The second or complete problem includes the determination of numbers which not only satisfy equations (1) to (7), but also

$$W + B = \text{a square number}, \tag{8}$$
$$D + Y = \text{a triangular number}, \tag{9}$$

and this is to be done by finding an integer $N$ to multiply into each of the results of the first problem, or

$$\begin{aligned} 17\,826\,996\,N &= \text{a square number}, \\ 11\,507\,447\,N &= \text{a triangular number}. \end{aligned}$$

A number $N$ that will satisfy one of these conditions can be found without difficulty but to determine $N$ so that both conditions will be satisfied is a task involving an enormous amount of time and labor. In fact, this required number $N$ has never been completely computed.

A solution which satisfies (8) as well as (1) to (7) is easily made. Since $W + B$ is $17\,826\,966\,N$ or $4 \times 4\,456\,749\,N$, and since 4 456 749 contains no number that is a perfect square, it is plain that $N$ must be 4 456 749. Accordingly, each of the numbers found in the first solution must be multiplied by this value of $N$ in order to satisfy (1) to (8) inclusive; the number $W + B$ is then 79 450 446 596 004

which is a perfect square, but $D + Y$ is 51 285 802 909 803 which is not a triangular number.

It is now time to explain what is meant by a triangular number. The number ten is triangular because ten dots can be arranged in rows in the form of a triangle, the first row having one dot, the second two, the third three, and the fourth four dots. The next higher triangular number is 15 and the next 21, and in general $\frac{1}{2} n (n + 1)$ is a triangular number whenever $n$ is an integer, $n$ being the number of rows in the triangle. The number 51 285 802 909 803 is shown not to be a triangular number by equating it to $\frac{1}{2} n (n + 1)$ and computing $n$ from the quadratic equation thus formed when it is found that $n$ is not an integer.

Now since 51 285 802 909 803 is the number of yellow and dappled bulls which results from a solution which satisfies equations (1) to (8) inclusive, it is plain that the ninth condition may be expressed by

$$51\ 285\ 802\ 909\ 803\, x^2 = \tfrac{1}{2} n (n + 1),$$

in which $x$ and $n$ are to be integers. When $x^2$ has been found each of the numbers of the first solution is to be multiplied by $4\,456\,749\, x^2$ in order to give the number of bulls and cows in each herd which satisfy the nine imposed conditions.

These numbers were readily seen to be so great that the island of Sicily could not contain all the cattle the problem seems to demand. This requirement, however, was understood to be only figurative, and mathematicians agreed that the numbers might be found altho no useful purpose would be attained by computing them. Thus the question rested until 1860 when Amthor demonstrated that 206 545 figures are necessary to express the total number of

cattle and that $766 \times 10^{206542}$ gives their approximate number. This is an enormous number, and it is not difficult to show that a sphere having the diameter of the milky way, across which light takes ten thousand years to travel, could contain only a part of this great number of animals, even if the size of each is that of the smallest bacterium.

It would be thought that, after this investigation of Amthor, the cattle problem would have been finally dropped but such was not the case. The way to solve it was well understood from the theory of indeterminate analysis. Let the preceding equation be multiplied by 8, unity be added to each member and let $2n + 1$ be called $y$; then it becomes

$$y^2 - 410\,286\,423\,278\,424\,x^2 = 1$$

which is of the form $y^2 - Ax^2 = 1$, and it is known that when $A$ is an integer there can always be found integral values of $x$ and $y$ which satisfy the equation. The method of solution cannot well be explained here, but it was devised many years ago by Pell and Fermat and is well known to those skilled in higher mathematics. For example, take the simple case where $A = 19$, or $y^2 - 19x^2 = 1$, then the smallest integral values of $y$ and $x$ are 170 and 39.

In 1889 A. H. Bell, a surveyor and civil engineer of Hillsboro, Illinois, began the work of solution. He formed the Hillsboro Mathematical Club, consisting of Edmund Fish, George H. Richards, and himself, and nearly four years were spent on the work. They computed thirty of the left-hand figures and twelve of the right-hand figures of the value of $x$ without finding the intermediate ones. This value is $x = 34\,555\,906\,354\,559\,370 \ldots 252\,058\,980\,100$ in which the dots indicate fifteen computed figures, which it is unnecessary to give here, and 206 487 uncomputed

ones; the total number of figures in this number is 206 531. The final step is to multiply each of the numbers of the first solution by 4 456 749 and by this value of $x^2$, thus giving:

| | | |
|---|---|---|
| White bulls | = 1 596 510 ..... | 341 800 |
| Black bulls | = 1 148 971 ..... | 178 600 |
| Dappled bulls | = 1 133 192 ..... | 894 000 |
| Yellow bulls | = 639 034 ..... | 026 300 |
| White cows | = 1 109 829 ..... | 564 000 |
| Black cows | = 735 594 ..... | 645 400 |
| Dappled cows | = 541 460 ..... | 318 000 |
| Yellow cows | = 837 676 ..... | 113 700 |
| Total cattle | = 7 760 271 ..... | 081 800 |

in which each line of dots represents 206 532 figures, the total number of figures in each line being either 206 545 or 206 544. In each of these lines there are omitted twenty-four figures at the left end and six at the right end which were computed by the Hillsboro Mathematical Club.

This solution is published in the American Mathematical Monthly for May, 1895, where Bell remarks that each of these enormous numbers is "one-half mile long." A clearer idea of its length may be obtained by considering the space required to print it. Each page of this volume contains 32 lines and in each line about 50 figures may be printed, so that one page could contain about 1750 figures. To print a number of 206 245 figures would require 129 pages, and to print the nine numbers indicated above a volume of over 1000 pages would be needed.

It is known that Archimedes speculated regarding large numbers, for his book *Arenarius* is devoted to showing that a number may be written that will express the number of grains of sand in a sphere of the size of the earth. It cannot be proved that Archimedes was the author of the cattle problem, but as Amthor remarks, the enormous

numbers in its solution render it worthy of his genius and proper to bear his name. Its closing challenge still remains open, for the complete solution has not yet been made; and the investigations of Bell show that this would require the work of a thousand men for a thousand years. The little prairie town of Hillsboro, may, however, well exult as a conqueror, for its mathematical club has made the most complete of all solutions of the cattle problem and has proved itself to be highly skilled in numbers.

## 55

## MAGIC SQUARES

Fig. 4 shows the well-known magic square containing the nine digits, the sum of each row, column, and diagonal being 15. These numbers may be arranged in other ways, for instance, by taking the left-hand column as the top row, the middle column as the middle row, and the right-hand column as the lowest row. Altogether there are eight different arrangements for this simplest of all magic squares.

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

Fig. 4

| | | | |
|---|---|---|---|
| 1 | 14 | 15 | 4 |
| 12 | 7 | 6 | 9 |
| 8 | 11 | 10 | 5 |
| 13 | 2 | 3 | 16 |

Fig. 5

| | | | |
|---|---|---|---|
| 16 | 2 | 11 | 5 |
| 3 | 13 | 8 | 10 |
| 6 | 12 | 1 | 15 |
| 9 | 7 | 14 | 4 |

Fig. 6

A magic square of sixteen numbers is shown in Fig. 5, the sum of each row, column, and diagonal being 34. Here also there are many other arrangements. Fig. 6 shows another magic square which has the same properties and there are others which will be explained later.

A true magic square should have 1 for its smallest number and contain all the natural numbers up to $n^2$, where $n$ is the

number of cells in one row or column of the square. The sum of the first $n^2$ natural numbers is $S = \frac{1}{2} n^2 (n^2 + 1)$, and hence the sum of the numbers in one row, column, or diagonal is $N = \frac{1}{2} n (n^2 + 1)$. Accordingly, for magic squares of 9, 16, 25, 36, 49, 64 cells the fundamental data are:

| Cells in one row | $n =$ 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| Sum of all numbers | $S =$ 45 | 136 | 325 | 666 | 1225 | 2080 |
| Sum of one row | $N =$ 15 | 34 | 65 | 111 | 175 | 260 |

A magic square with 25 cells has the sum of the numbers in each row, column, or diagonal equal to 65. To form such a square write the numbers 1, 2, 3, 4, 5 in each row of the square in Fig. 7 so that the mean number 3 always comes

| | | | | |
|---|---|---|---|---|
| 3 | 4 | 5 | 1 | 2 |
| 2 | 3 | 4 | 5 | 1 |
| 1 | 2 | 3 | 4 | 5 |
| 5 | 1 | 2 | 3 | 4 |
| 4 | 5 | 1 | 2 | 3 |

Fig. 7

| | | | | |
|---|---|---|---|---|
| 15 | 20 | 0 | 5 | 10 |
| 20 | 0 | 5 | 10 | 15 |
| 0 | 5 | 10 | 15 | 20 |
| 5 | 10 | 15 | 20 | 0 |
| 10 | 15 | 20 | 0 | 5 |

Fig. 8

| | | | | |
|---|---|---|---|---|
| 18 | 24 | 5 | 6 | 12 |
| 22 | 3 | 9 | 15 | 16 |
| 1 | 7 | 13 | 19 | 25 |
| 10 | 11 | 17 | 23 | 4 |
| 14 | 20 | 21 | 2 | 8 |

Fig. 9

in one of the diagonals, then write the numbers 0, 5, 10, 15, 20 in the rows of Fig. 8 so that the mean number 10 always comes in the opposite diagonal. For Fig. 7 the sum of each row, column, and diagonal is 15 and for Fig. 8 it is 50, the total being 65. Then add the numbers in corresponding cells and put the results in Fig. 9 thus forming a magic square of 25 cells where the sum of each row, column, and diagonal is 65.

Another magic square of 25 cells may be formed by taking the first five natural numbers in the order 4, 1, 3, 2, 5 and placing them in each row of a square in this order, 3 always coming in a diagonal cell; then in another square writing 15, 0, 10, 5, 20 so that 10 comes in each cell of the

opposite diagonal; and finally adding the numbers in corresponding cells. Other squares may be formed by writing the first five natural numbers in different orders, keeping always 3 in the middle, and arranging the auxiliary numbers correspondingly. Altogether twelve magic squares may be formed in this way, and from each of these still others may be formed by interchanging rows and columns.

Any magic square with an odd number of cells $n$ in one row, can be formed in a similar way, by writing the first $n$ natural numbers in each row so that $\frac{1}{2}(n+1)$ comes in a diagonal cell, then writing the numbers $0, n, 2n, 3n, \ldots (n-1)n$, so that $\frac{1}{2}(n-1)n$ comes in the cells of the opposite diagonal, and finally adding the numbers in corresponding cells. Thus for $n = 7$, the first set of numbers might be 5, 3, 1, 4, 6, 2, 7, when the second set must be 28, 14, 0, 21, 35, 7, 42; here 4 must be written along one diagonal and 21 along the other diagonal.

The formation of magic squares having an even number of cells is not so easy and it seems that a general rule has not been given. For 16 cells, however, the following rule is applicable. Write in the upper and lower rows of a square the numbers 1, 3, 2, 4; then in the two middle rows write them in the reverse order. Again in the top row put the numbers 0, 12, 12, 0, and in the lowest row write 12, 0, 0, 12; in the upper middle row put 8, 4, 4, 8 and in the one below it 4, 8, 8, 4. The addition of these numbers will give a magic square of 16 numbers which will be slightly different from those shown in Figs. 5 and 6.

A magic square of 64 cells may be seen on page 163 of Ball's Mathematical Recreations and Essays (London, 1911), which possesses the wonderful property that if each number be replaced by its square the resulting square is

also magic, the sum of the numbers in each line being 11 180.

We shall now briefly mention the magic squares sometimes called "diabolic," or more commonly "Nasik," this being the name of the town in India where A. H. Frost invented them. Fig. 6 shows a Nasik magic square of 16 cells where the sum of each row, column, and diagonal is 34. The sum of the numbers in each broken diagonal is also 34, a broken diagonal being one which is partly on one side of the main diagonal and partly on the other side; thus the numbers 11, 13, 6, and the number 4 constitute a broken diagonal, as likewise do the numbers 2, 3, and 15, 14. In this magic square the sum of the numbers in any small square formed by four adjacent cells is also equal to 34. Truly, this is a marvellous arrangement of the first sixteen natural numbers.

Benjamin Franklin, the famous philosopher and diplomat, amused himself with magic squares. At page 251 of Volume 3 of his Collected Works (Philadelphia, 1808) may be seen a square having eight cells on a side, or 64 cells in all, in which the sum of each row and column is 260, while the sum of the numbers in any four adjacent cells is 130. This, however, is not a true magic square, as the sum of the numbers is 292 for one diagonal and 228 for the other. Franklin also devised a square of 2056 cells which is called the "magic square of squares," and a magic circle having many curious properties.

Squares having the sum of the numbers in each line greater than $\frac{1}{2} n (n^2 + 1)$ may be formed by adding an integer $I$ to each number so that the sum of all the numbers in each line is $N = \frac{1}{2} n (n^2 + 1) + nI$. When $N$ is given, values of $n$ may sometimes be found which satisfy

this equation. Thus for $N = 1000$ and $n = 4$ the value of $I$ is integral, namely, 187, so that if 187 be added to each of the numbers in Fig. 6, the resulting numbers have the property that the sum of each row, column, and diagonal is 1000. The greatest and least numbers in such a square may be found from the expression $N/n \pm \frac{1}{2}(n^2 - 1)$; thus for $N = 1000$ and $n = 5$ they are 212 and 188.

Among other curious squares are those which are filled with the first $n^2$ natural numbers by the knight's move in chess, each square being occupied only once by the knight. Leonard Euler, the great mathematician, amused himself with such squares and two which he constructed for squares of five and seven sides may be seen in the Encyclopedia Britannica; these squares, however, are not magic, although they have certain curious properties. It is well known that this problem may be solved on the common chess board in many different ways; and one of these gives a square which is magic.

## CHAPTER III

## GEOMETRY

### 56

HE QUEEN of mathematics is the ancient geometry as exemplified by Euclid. Elegant, chaste, and beautiful is its logic, wonderful are its conclusions. It originated in Egypt and came to its development at the great university of Alexandria where Euclid was the founder of its mathematical school. The Greek words $\gamma\eta$ and $\mu\epsilon\tau\rho o\nu$, which form the name of the science, mean land and measure, respectively, so that geometry was originally the measurement of land. In Egypt where the annual inundations of the Nile easily obliterate the boundaries of parcels of land, perhaps the rules of geometry received their first practical application.

### 57

Euclid lived about 300 B. C. Tradition says that he was mild and unpretending in manner and kind to all genuine students of mathematics. On one occasion a student complained that geometry brought no profit, whereupon Euclid directed that three oboles be given him. To King Ptolemy, who asked if there was not an easier way to understand geometry than through study of the Elements, Euclid gave the reply "there is no royal road to geometry."

### 58

The Elements of Euclid is the title given to that presentation of plane geometry which Euclid prepared about 300

B. C. Written in Greek, it later was translated into Arabic, then into Latin. Translations have appeared in all European languages which have been annotated by many different editors. More than a thousand editions have appeared since the invention of printing about 1480; in fact, no book except the Bible has passed through so many editions as the Elements of Euclid.

The first six books of Euclid were generally used in colleges and schools in England and America until about 1860. The logic of the presentation is generally perfect, and the treatise gives important facts of plane geometry. It can be successfully used today, for it is certain that it is much better than some texts now on the market and equally as good as many of them.

## GEOMETRIC AMUSEMENTS

### 59

The second proposition of the first book of Euclid affords amusement to some beginners because it appears to them that a much simpler method might have been used. The first proposition is to construct an equilateral triangle upon a straight line of given length. The second is "to draw from a given point a line equal to a given straight line"; to do this Euclid lets $A$ be the given point and $BC$ the given straight line; then joining $A$ and $B$ he constructs upon $AB$ the equilateral triangle $ABD$ by the method of the first proposition. From $B$ as a center with a radius $BC$ the circle $CC'$ is described and $DB$ is produced until it meets the circle in $E$. Then from $D$ as a center with the radius $DE$ a

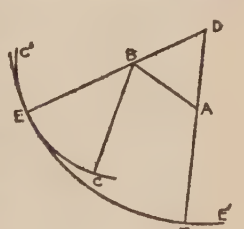


Fig. 10

second circle $EE'$ is described and $DA$ is produced until it meets the circle in $F$. Accordingly, from the given point $A$ the straight line $AF$ has been drawn equal to the given straight line $BC$. The reader can easily supply the steps of the demonstration, but can he state why Euclid did not at once describe from the center $A$ a circle with the radius $BC$?

60

The fifth proposition of the first book of Euclid has always been known as the "pons asinorum," and it has been generally implied that the boy who failed to understand it was an ass. The word "pons" or "bridge" perhaps originated from the figure which roughly resembles the rude bridge truss used to span narrow streams.

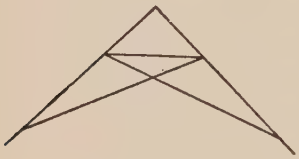


Fig. 11

61

The most important proposition in the first book of Euclid's Elements of Geometry is the forty-seventh, namely: The square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. This truth was known to Hindoos and Egyptians long before the time of Euclid, but Pythagoras, who lived 550 B. C., gave a formal demonstration which has caused his name to be frequently applied to the theorem.

62

There are many right-angled triangles having the three sides expressed by integral numbers. The simplest one has 3, 4, 5 for its sides and this was known to the Egyptians who

used it many centuries before the Christian era in constructing the pyramid of Cephren at Gizeh. If each of these sides be doubled we have 6, 8, 10 for the sides of a right-angled triangle which has probably been known to surveyors in all ages and which is now constantly used by them in laying off a right angle with the chain. The following are all the right-angled triangles with sides expressed in integers for which the shortest side does not exceed 15:

$3^2 + 4^2 = 5^2$
$5^2 + 12^2 = 13^2$
$6^2 + 8^2 = 10^2$
$7^2 + 24^2 = 25^2$
$8^2 + 15^2 = 17^2$
$9^2 + 12^2 = 15^2$
$9^2 + 40^2 = 41^2$
$10^2 + 24^2 = 26^2$
$11^2 + 60^2 = 61^2$
$12^2 + 16^2 = 20^2$
$12^2 + 35^2 = 37^2$
$13^2 + 84^2 = 85^2$
$14^2 + 48^2 = 50^2$
$15^2 + 20^2 = 25^2$
$15^2 + 36^2 = 39^2$

### 63

Let $m$ and $n$ be any two integers; then if $2mn$ be taken for one of the legs of a right-angled triangle, the other leg is $m^2 - n^2$ and the hypothenuse is $m^2 + n^2$. Thus let $m = 9$ and $n = 4$, then the three sides are 72, 65, 97. Let the reader use this rule to determine three right-angled triangles each having 48 for one of its legs.

### 64

The following right-angled triangles with integral sides have the same area: first triangle, 24, 70, 74; second triangle, 40, 42, 58; third triangle, 15, 112, 113.

### 65

A castle wall there was, whose height was found
To be just fifty feet from top to ground;

Against the wall a ladder stood upright,
Of the same length the castle was in height.
A waggish fellow did the ladder slide,
(The bottom of it) five feet from the side.
Now I would know how far the top did fall
By pulling out the ladder from the wall?

### 66

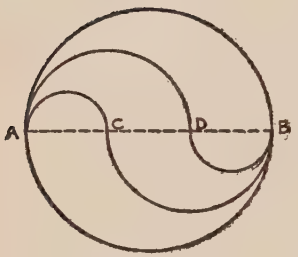


Fig. 12

To divide a circle into three equal parts. One method is as follows: Divide a diameter $AB$ into three equal parts $AC$, $CD$, $DB$; on $AC$ and $CB$ describe the semi-circles shown, also on $AD$ and $DB$ describe semi-circles; then is the area of the given circle divided into three equal parts.

### 67

When is the sum of the squares of two successive integers a perfect square? This is answered by Osborne in American Mathematical Monthly for May, 1914. The two smallest are $0^2 + 1^2 = 1^2$ and $3^2 + 4^2 = 5^2$. Then come $20^2 + 21^2 = 29^2$ and $119^2 + 120^2 = 169^2$. Five larger sets are also found the largest of which is $803\,760^2 + 803\,761^2 = 1\,136\,689^2$.

### 68

To divide a circle into $n$ equal parts an approximate solution is the following: Let $AB$ be the diameter of the given circle and $CD$ a diameter perpendicular to $AB$. Divide $AB$ into $n$ equal parts. On the diameter $DC$ produced lay off $CE$ equal to one-third of the diameter. From $E$ draw a straight line through the second point of division

on $AB$ and produce it to meet the circumference in $F$. Then the arc $AF$ is the $n$th part, very closely, of the circumference.

## 69

The trisection of the angle is one of the famous problems of antiquity. It can be solved in many ways, but not by plane geometry, for Euclid allowed no instruments but a straight ruler and the compasses. However, if it be permitted to mark or graduate the ruler, the problem can be solved in the following manner, as was first shown by Archimedes. Let $BAC$ be the angle to be trisected and from $A$ as a center describe a semi-circle with the compasses. Produce the radius $BA$ toward the right. From one end of the ruler lay off on it a distance equal to the radius $AC$ and mark the point thus found. Place the ruler so that one edge coincides with $C$ while the end moves along the produced line $BD$. When the mark on the ruler coincides with the semi-circle put there a point $E$, then the arc $EF$ is one-third of the arc $BC$ and the angle $EAF$ is one-third of the given angle $BAC$. The reader can easily supply the demonstration by considering the dots that have been placed on the figure.

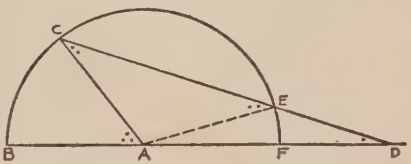


Fig. 13

## 70

### THE VALUE OF $\pi$

Archimedes deduced, about 220 B. C., a rule for the quadrature of the circle, proving that its area is equal to

the square of its radius multiplied by a number which lies between 3 10/71 and 3 10/70. Or, area = $\pi r^2$, where $r$ is the radius and $\pi$ is the number whose approximate value is 3 1/7. This number $\pi$ is also the ratio of the circumference of the circle to its diameter and it turns up in connection with many problems not at all related to the circle.

### 71

The value of $\pi$ to four decimal places is 3.1416. "Yes, I have a number," is a sentence in which the number of letters in each word corresponds to the integers in this value of $\pi$. The following, which appeared in the Scientific American of March 21, 1914, will enable its value to be remembered to 12 decimals:

> See I have a rhyme assisting
> My feeble brain its tasks sometimes resisting.

### 72

An early statement regarding the ratio of the circumference of a circle to its diameter is found in the Bible in connection with the description of Solomon's temple. The architect of this magnificent building was Hiram, a widow's son, whose father was a man of Tyre. In I Kings, vii, 23, and also in II Chronicles, iv, 2, we find the dimensions of a circular tank or pond which was designed by Hiram. "He made a molten sea, ten cubits from one brim to the other; it was round all about and its height was five cubits; and a line of thirty cubits did compass it about." It should not be inferred from this description, however, that the value $\pi = 3$ was used in computations by this distinguished architect. The date of the construction of Solomon's temple was about 1007 B. C.

73

The early Romans are said to have used 3 1/8 for the value of $\pi$ but Frontinus, in 97 A. D., used 3 1/7, as is seen from the list of circumferences and diameters of water pipes which is mentioned in No. 3. He also used this value in computing areas, as appears from his statement: "The square digit is greater than the round digit by three-fourteenths of itself; the round digit is smaller than the square digit by three-elevenths."

The value of $\pi$ was computed by William Shanks in 1873 to 707 decimal places, surely a great waste of labor, for the most refined computation requires only seven or eight decimals, and in all usual work 3.1416 is close enough. It has been proved that the number $\pi$ is incommensurable, that is, the number of its decimals is infinite.

## THE PYRAMIDS OF EGYPT

74

It has been claimed that the great pyramid at Gizeh in Egypt was intended to be so built that the length of the four sides of the base should be the circumference of a circle whose radius was the vertical height. Petrie's measurements of 1882 give 9068.8 inches for the length of one side of the base and 5776.0 inches for the height of the pyramid when its sides met at an apex. Now $9068.8/5776.0 = 1.5703$, whereas $\frac{1}{2}\pi$ is 1.5708. Probably they used 3 1/7 for the value of $\pi$; in this case the ratio of one of the sides of the base to the height would have been 11/7 or 1.5714. Petrie's measures of the angle made by the sides of the pyramid with the horizontal gave the mean value $51^\circ\ 52'$; this corresponds to 1.5735 for the above ratio.

The pyramid of Cephren at Gizeh has its sides inclined to the horizontal at an angle of 53° 10′. This corresponds very closely to a slope of 4 on 3, so that the right-angled triangle having sides of 3, 4, 5 seems to have been used in its construction. Here the ratio of one side of the base to the height is $6/4 = 1.5$ instead of 1.57 as in the other pyramids at Gizeh. Undoubtedly mathematics and astrology controlled the design of these pyramids, altho their final purpose was for tombs for the kings. So mighty is the great pyramid at Gizeh and so solidly is it constructed that it will undoubtedly remain standing long after all other buildings now on the earth have disappeared.

### 75

Herodotus said that the area of an inclined face of the pyramid was equal to a square described upon its altitude. What value does this condition give for the angle which the plane of a face makes with the base?

### 76

The King's Chamber in the great pyramid is 10 cubits wide, 20 cubits long, and 11.18 cubits high. These figures result from Petrie's measurements made in English inches, 20.612 inches being taken for the length of the old Egyptian cubit. The height was hence made one-half of the floor diagonal, so that the three dimensions of the room are 10, 20, $\frac{1}{2}\sqrt{500}$ cubits, and the solid diagonal is 25 cubits in length. These numbers are proportional to 1, 2, $\frac{1}{2}\sqrt{5}$, 2.5 It can hardly be supposed that these dimensions were accidental; they were probably introduced into the design in accordance with some astrological superstition of a mathematical nature.

77

A magnitude or quantity is anything that can be measured. Can a solid angle, like that at the apex of a pyramid, be measured? No one ever spoke of a solid angle as being twice as large as another one. The only measure of a solid angle that has been proposed is the surface of a sphere described from its apex and included between its sides. The radius of the sphere being taken as unity, $\frac{1}{4}\pi$ would be the measure of the solid angle at one corner of a cube. What is the measure of the solid angle at the apex of a right cone whose altitude is equal to the radius of its base?

78

THE PRISMOIDAL FORMULA

A general method of finding the volume of any of the solids of common geometry is the Prismoidal Formula. Let $A$ and $B$ be the areas of the two parallel bases and $C$ the are aof a parallel section halfway between them; let $h$ be the altitude between the bases $A$ and $B$. Then the volume of the solid is $V = 1/6\, h\,(A + 4\,C + B)$.

To apply this to a cone which has a base of radius $r$ and the altitude $h$, the upper base $A$ is 0, since it is at the apex of the cone, the lower base $B$ is $\pi r^2$, and $C$, the area of a section halfway between the two bases is $\frac{1}{4}\pi r^2$. Then the volume is $V = 1/6\, h\,(0 + \pi r^2 + \pi r^2) = 1/3\,\pi r^2 h = 1/3\,Bh$.

To find the volume of a sphere draw two parallel planes tangent to it, giving the two bases $A = 0$ and $B = 0$; the area of a section halfway between them is $\pi r^2$, where $r$ is the radius of the sphere; also the altitude $h$ is $2\,r$. Then volume $= 1/6\,(2\,r)\,(0 + 4\,\pi r^2 + 0) = 4/3\,\pi r^3$.

To find the volume of a masonry pier 16 feet high, the

top $B$ being a rectangle $8 \times 24$ feet, and the lower base being a rectangle $12 \times 30$ feet inside. The areas of the bases are 192 and 360 square feet. The dimensions of a section $C$ halfway between the bases are $\frac{1}{2}(8 + 12)$ or 10 feet and $\frac{1}{2}(24 + 30)$ or 27 feet, so that the area of $C$ is 270 square feet. Then

$$\text{Volume} = 1/6 \times 16\,(360 + 1080 + 192) = 4352 \text{ cu. ft.}$$

This is a problem which is difficult to solve by the methods of common geometry, for the sides of the pier when produced do not meet at a point, and hence the rule for a truncated pyramid does not apply.

The prismoidal formula gives volumes of the ellipsoid, paraboloid, and other solids generated by the revolution of curves of the second and third degree about an axis. It also applies to warped surfaces like the hyperbolic paraboloid when the areas $A$, $B$, $C$ are known or can be found.

Let the student apply the prismoidal formula to find the volume of a segment of a sphere whose altitude is $h$ and the radius of whose base is $a$. Here a little analytic geometry is perhaps necessary to find $C$ in terms of $a$ and the radius $r$ of the sphere.

## 79

### GEOMETRIC FALLACIES

To prove, geometrically, that 24 equals 25; draw a square on cardboard, 5 inches on a side, having an area of 25 square inches, as shown in Fig. 14; then cut the cardboard into four pieces as indicated by the three broken lines; these four pieces can then be arranged in the rectangular form shown in Fig. 15, where there are three inches on one side and eight on the other, giving twenty-four square inches in

all. Hence it has been proved geometrically that 24 equals 25. Where is the fallacy?

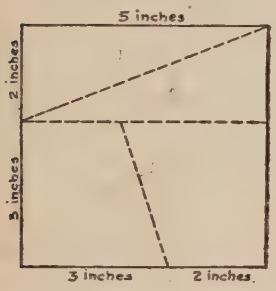



Fig. 14

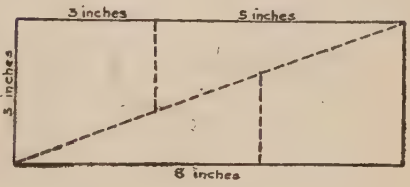



Fig. 15

80

To prove that a straight line can be divided into four parts so that the first point is to the third as the third is to the fifth. Let $AE$ be the given straight line which is so divided that $AB : BC :: CD : DE$; then $AB/BC = CD/DE$; now cancelling $B$ out of the first member of this equation and $D$ out of the second, there results $A/C = C/E$, or $A : C :: C : E$, which proves the proposition enunciated.

81

The semicircumference of a circle is equal to its diameter. Let the diameter be divided into four equal parts and on each part let a semicircle be described. These four smaller semicircles are equal to the given semicircumference; for let $d$ be the given diameter, then $\frac{1}{2}\ \pi d$ is the corresponding semicircumference; each of the equal parts of the diameter is $\frac{1}{4}\ d$ and the corresponding semicircumference is $1/8\ \pi d$; hence the sum of these four small semicircumferences is $4\ (1/8\ \pi d)$ or $\frac{1}{2}\ \pi d$. Now divide each of the parts of the diameter into four equal parts and describe semicircles on

them and it is clear that the sum of the sixteen semicircumferences is equal to the large semicircumference originally given. Thus continue, and when the number of points of division is infinite the sum of all the infinitely small circumferences is equal to the large original one. But when this occurs all the small semicircumferences coincide with the diameter of the circle and their sum is hence equal to it. Accordingly, it has been proved that the semicircumference of any circle is equal to its diameter. Where lies the fallacy?

## 82

To prove that any obtuse angle is equal to a right angle. Let *ABC* be an obtuse angle and *DCB* a right angle; it is required to prove that these angles are equal. Make *CD* equal to *BA*, join *AD*, bisect it in *E* and draw *EG* perpendicular to *AD*. Also bisect *BC* in *F* and draw *FG* perpendicular to *BC*. The two perpendiculars meet in *G*. Draw *GA*, *GD*, *GB*, and *GC*; then *GA* equals *GD*, and *GB* equals *GC*. Since also *CD* = *BA*, the sides of the triangle *GBA* are equal, each to each, to the sides of the triangle *GCD*. Hence these triangles are in every way equal; and the angle opposite the side *GA* is equal to the angle opposite the side *GD*. Accordingly the angle *ABG* equals the angle *DCG*; subtracting from these the equal angles *CBG* and *BCG*, the result is that the angle *ABC* is equal to the angle *DCB*. Therefore, it has been proved that the obtuse angle *ABC* is equal to the right angle *DCB*. The fallacy in this demonstration is not easy to detect, but nevertheless it is there.

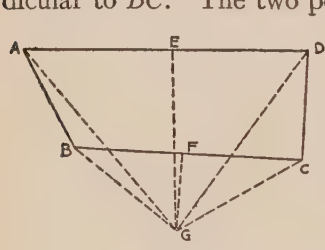


Fig. 16

## 83

The circumference of a small circle is equal to the circumference of a larger circle. Let a wheel roll along a horizontal plane until it has made one revolution; then the line $AB$ is equal to its circumference. The small circle in

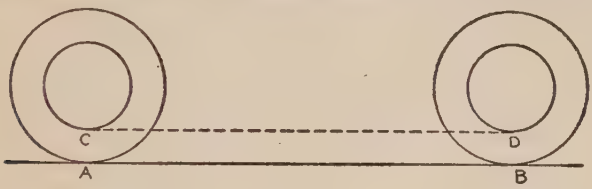



Fig. 17

the figure has also in the same time made one revolution, since it is drawn on the side of the wheel concentric with the larger circle. Hence the circumference of the small circle rolls out the line $CD$ which is equal to $AB$. Therefore, the two circumferences are equal. Where is the fallacy?

## 84

Every triangle is isosceles, or two angles of any triangle are equal to each other. Let $abc$ be any triangle. Bisect the angle $a$; from the middle of $bc$ draw a normal to $bc$; the bisector and the normal meet at a point. From this point draw lines to $b$ and $c$ and normals to the sides $ab$ and $ac$. The triangles $C$ and $D$ are then equal;

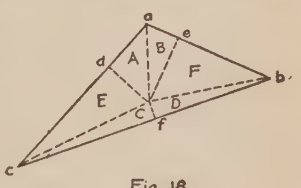



Fig. 18

also the triangles $A$ and $B$ are equal, whence $ad = ae$. Accordingly, the third pair of triangles $E$ and $F$ must be equal, whence $cd = be$. Hence $ad + cd = ae + eb$, or $ab = ac$. Thus it has been proved that any triangle is isosceles.

## 85

The three following propositions are certainly interesting. Are they true or false? (1) Let $AB$ be a straight line and $C$ any point on it. On $AC$ and $BC$ as bases construct the isosceles triangles $AbC$ and $BaC$ so that the equal sides make angles of 30° with the bases. Also on $AB$ construct the isosceles triangle $AcB$ so that the equal sides make angles of 30° with $AB$. Draw $ab$, $bc$, $ca$ as shown by the

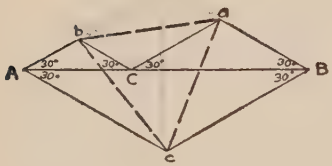



Fig. 19

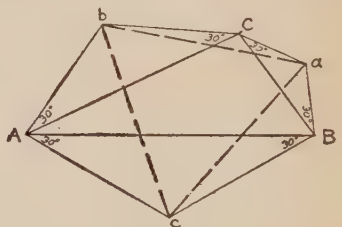



Fig. 20

broken lines. Then each of the angles of the triangle $abc$ is 60°. (2) Let $ABC$ be any triangle. On the sides as bases construct the isosceles triangles $AcB$, $BcA$, $CbA$, so that the equal sides of each make angles of 30° with its base. Draw $ab$, $bc$, $ca$, as shown by the broken lines. Then each of the angles of the triangle $abc$ is 60°. (3) Describe a square on each of the sides of a right-angled triangle. At the centers of these squares put the points $a$, $b$, $c$, and join these points so as to form the triangle $abc$. Then each of the angles of this triangle is 60°.

## 86

The following remarkable fallacy appeared in the *Forum* of April, 1914: "A cube will readily present to the eye three dimensions. length, height, and breadth. Four

diagonal lines imagined from the corners of the cube will each be at right angles to the other three; hence we have four dimensions. We should find it difficult to construct anything along the lines of these four dimensions for the simple reason that the work would have to begin at the point where the lines intersect and progress outward through within the four lines. We might call these four lines expansion boundaries for if you would cause a cube to expand and maintain its symmetry or proportions, it would expand along these four lines. Any solid can therefore be considered a cross section of its greater self. The foregoing is the only practical demonstration that can be given of four dimensions."

## 87

## ON THE AREA OF A CLOSED TANGLE

From Clifford's *Common Sense of the Exact Sciences*, Fifth Edition (London, 1907), pages 135–137.

Hitherto we have supposed the areas we have talked about to be bounded by a single loop. It is easy, however,

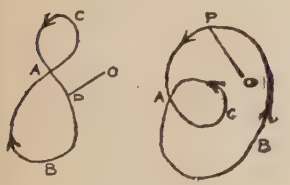



Fig. 21

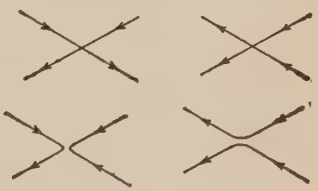


Fig. 22

to determine the area of a combination of loops. Thus, consider the figure of eight in Fig. 21 which has two loops; if we go around it continuously in the direction indicated

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by the arrowheads, one of these loops will have a positive, the other a negative area, and therefore the total area will be their difference, or zero if they be equal. When a closed curve, like a figure of eight, cuts itself it is termed a *tangle*, and the points where it cuts itself are called *knots*. Thus a figure of eight is a tangle of one knot. In tracing out the area of a closed curve by means of a line drawn from a fixed point to a point moving around the curve, the area may vary according to the direction and the route by which we suppose the curve to be described. If, however, we suppose the curve to be sketched out by the moving point, then its area will be perfectly definite for that particular description of its perimeter.

We shall now show how the most complex tangle may be split up into simple loops and its whole area determined from the areas of its simple loops. We shall suppose arrowheads to denote the direction in which the perimeter is to be taken. Consider either of the accompanying figures (Fig. 21). The moving line $OP$ will trace out exactly the same area if we suppose it not to cross the knot at $A$ but first trace out the loop $AC$ and then to trace out the loop $AB$ in both these cases going around these two loops in the direction indicated by the arrowheads. We are thus able in all cases to convert one line cutting itself in a knot into two lines, each bounding a separate loop, which just touch at the point indicated by the former knot. This dissolution of knots may be suggested to the reader by leaving a vacant space where the boundaries of the loops really meet. The two knots in Fig. 22 are shown dissolved in this fashion.

The reader will now have no difficulty in separating the most complex tangle into simple loops. The positive or negative character of the areas of these loops will be suffi-

ciently indicated by the arrowheads on their perimeters. We append an example (Fig. 23).

Fig. 23

In this case the tangle reduces to a negative loop $a$ and to a large positive loop $b$, within which are two other positive loops $c$ and $d$, the former of which contains a fifth small positive loop $e$. The area of the entire tangle then equals $b + c + d + e - a$. The space marked $s$ in the first figure will be seen from the second to be no part of the area of the tangle at all.

## 88

## MAP COLORING

It has long been known that only four different colors are necessary in order to color the most complicated map of a country so that contiguous sides of districts shall not have the same color. About 1850 this fact was brought to the attention of mathematicians but, altho much discussion of it has been made, a rigorous proof that only four colors are necessary has not yet been made.

Fig. 24 shows a map of nine districts in which the four colors $A$, $B$, $C$, $D$ are used for eight of the areas and there may seem no way to use one of these colors for the other district unless it adjoins upon the same color. However, by the very slight change shown in Fig. 25 the problem is

readily solved. Thus in all cases a way can be found to color the map by using only four different colors.

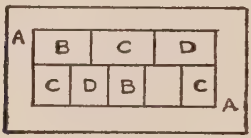



Fig. 24

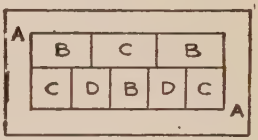



Fig. 25

The reason that five colors are not required seems to be that it is impossible to draw five areas so that a boundary of each shall be contiguous to the other four. Fig. 26 shows four areas each of which has its boundary contiguous

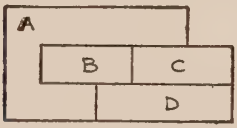



Fig. 26

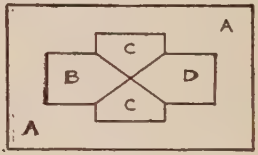



Fig. 27

with a boundary of the other three areas, but no way can be found to add a fifth area so that it may be contiguous to the other four. Four colors are sufficient for any map because no map has yet been drawn in which five areas are contiguous to four others. But no proof has yet been discovered that it is impossible to draw five such areas.

The word "contiguous" means that the areas border along a line, not at a point. Districts sometimes occur so that four or more of them meet at a point; for example, in Fig. 27 the two areas colored *C* meet at a point. Here more than four colors are needed if it is desired to have the areas on opposite sides of the point of junction different in shade.

# CHAPTER IV

## TRIGONOMETRY

### 89

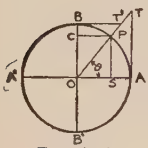
THE SOLUTION of triangles was the original object of Trigonometry, but it has been extended in modern times to include a vast realm of facts regarding functions of angles. The beginner in trigonometry is first introduced to the sine and cosine which are defined by a right-angled triangle in a manner essentially like the following: Let $a$ be the hypothenuse and $b$ and $c$ the legs, and $A$, $B$, $C$ the angles opposite to them, then the ratio $b/a$ is called the sine of the angle $B$, and the ratio $c/a$ is called the sine of $C$. Or as sometimes stated, the sine of an acute angle in a right-angled triangle is the ratio of the side opposite the angle to the hypothenuse. Thus $\sin B = b/a$ and $\sin C = c/a$.

Now it has been questioned by H. E. Licks whether this is the best way to define the sine for the beginner. The beginner is young and immature, to him the word "ratio" is more or less of an abstraction, and the fact that this ratio *is called* the sine does not appear to him significant. Why not state the definition something like this: The sine of an acute angle in a right-angled triangle is a number which multiplied by the hypothenuse gives the side opposite to the angle. This definition puts the matter in quite a different light for it gives the idea that the primary use of the sine is to solve a right-angled triangle, and it states the rule by

which one of the sides may be found when the sine of the opposite angle and the hypothenuse are known. How the sines are tabulated and used is a matter to be explained later.

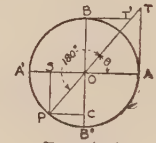
## 90

Fifty years ago a very different method of defining the trigonometric functions was in use. Let $AOP$ in Fig. 28 be an angle less than 90° which is measured from the line $AO$ around in a contrary direction to that of the hands of a watch. Let $P$ be any point on the quadrant $AB$ described with the radius $OA$ or $OP$; let $BB'$ be a diameter normal to $AA'$. From the point $P$ let perpendiculars $PS$ and $PC$ be

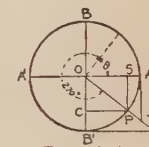


First Angle
Fig. 28

Second Angle
Fig. 29



Third Angle
Fig. 30



Fourth Angle
Fig. 31

drawn to the diameters $AA'$ and $BB'$. Then, if the radius $OA$ or $OP$ be of units length, $PS$ is the sine of the angle $AOP$ and $PC$ is its cosine. Also at $A$ let a perpendicular be drawn to $OA$ until it meets $OP$ produced, then $AT$ is the tangent of the angle $AOP$ and $OT$ is its secant; at $B$ let a perpendicular to the diameter $BB'$ be drawn until it meets $OP$ produced, then $BT'$ is the cotangent of the angle $AOP$ and $OT'$ is its cosecant.

Fig. 29 shows a similar representation for an angle $AOP$ when $P$ falls in the second quadrant, or for $90° + \theta$. Fig. 30 shows the functions for $180° + \theta$, and Fig. 31 for $270° + \theta$, the value of $\theta$ being in each case less than 90°. In each figure $PS$ and $PC$ are the sine and cosine of $AOP$, while $AT$

and $BT'$ are its tangent and cotangent and $OT$ and $OT'$ are its secant and cosecant.

Let the distances measured upward from $AA'$ be taken as positive and those measured downward be negative. Let distances to the right of $BB'$ be positive and to the left negative. Then the diagrams determine at once the signs of the trigonometric functions for the angles $AOP$ in the different quadrants. Thus when $P$ is in the first quadrant the angle $AOP$ or $\theta$ is less than $90°$, and its sine, cosine, and tangent are positive (Fig. 28). When $P$ is in the second quadrant it lies between $90°$ and $180°$, here the sine of $AOP$ is positive, while its cosine and tangent are negative (Fig. 29). For the third quadrant $AOP$ is between $180°$ and $270°$, its sine and cosine are negative but its tangent is positive (Fig. 30). For the fourth quadrant $AOP$ is between $270°$ and $360°$, its cosine is positive and its sine and tangent are negative (Fig. 31). The cotangent is seen to be positive for the first and third quadrants and negative for the second and fourth. The secant and cosecant are taken as positive when measured from $O$ toward $P$; hence they are both positive in the first and fourth quadrants, and both negative in the second and third.

The diagrams clearly show how each of the functions varies as an angle increases from $0°$ to $360°$. Take for example the tangent. When $\theta = 0$, $\tan\theta = 0$; as $\theta$ increases so does $\tan\theta$, when $\theta = 90°$, $\tan\theta = +\infty$; then just as $\theta$ passes through $90°$ $\tan\theta$ becomes $-\infty$; as $AOP$ increases towards $180°$ its tangent, though always negative, decreases numerically and becomes $0$ when $AOP = 180°$. In a similar manner the diagrams show that $\cos\theta$ is $1$ when $\theta = 0°$, decreases to $0$ when $\theta = 90°$, then becomes negative but increases numerically until its value

is $-1$ when $AOP$ is 180°. Thus the variation of each of the functions can be clearly traced.

These diagrams can be readily carried in the mind of a student after they have been intelligently constructed by him. They show instantly such relations as $\sin(90° + \theta) = \sin\theta$, $\cos(90° + \theta) = -\cos\theta$, $\tan(180° + \theta) = \tan\theta$, $\sin(270° + \theta) = -\sin\theta$, etc., $\theta$ being considered for this purpose as less than 90°. They also show that $\cos(90° - \theta) = \sin\theta$, $\cot(90° - \theta) = \tan\theta$, etc. Even in these modern days they may be useful to some of the beginners in trigonometry who have been brought up in a severer logic. At any rate the reason why the word tangent is used for one of the functions will be clear to him.

## 91

There are few paradoxes or fallacies in trigonometry, but here is one which is worth preserving. To prove that the angles of a triangle are equal to each other. Let $A$, $B$, $C$ be the angles of the triangle $ABC$, and $a$, $b$, $c$ the sides opposite to them. Produce $CA$ and make $AD$ equal to $c$, then the angles $ADB$ and $ABD$ are each equal to $\frac{1}{2}A$. Produce $BA$ and make $AE$ equal to $b$, then the angles $AEC$ and $ACE$ are each equal to $\frac{1}{2}A$. Now from the triangle $BCD$, there can be written

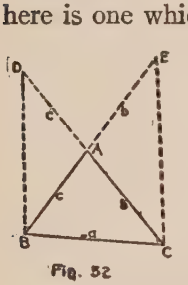


Fig. 52

$$\frac{CD}{BC} = \frac{b+c}{a} = \frac{\sin(B + \frac{1}{2}A)}{\sin\frac{1}{2}A}$$

and also from the triangle $CBE$,

$$\frac{BE}{BC} = \frac{b+c}{a} = \frac{\sin(C + \frac{1}{2}A)}{\sin\frac{1}{2}A}.$$

Equating these values of $(b + c)/a$ there results $\sin (B + \frac{1}{2} A) = \sin (C + \frac{1}{2} A)$. Therefore, $B = C$, and in like manner it may be shown that $A = B$. Hence the three angles of a triangle are equal to each other. Where lies the fallacy?

92

The computations of trigonometry are generally made by the help of logarithmic tables, but it is not wise that beginners should use logarithms at the start. Natural functions to three places are all that are needed in the blackboard work of a school and with such tables the angles of a triangle can be computed to the nearest quarter-degree from given sides. For example, when the two sides $a$ and $b$, and their included angle $C$ are given, the best formulas to use for finding the angles $B$ and $A$ are

$$\cot B = \frac{a/b}{\sin C} - \cot C, \quad \cot A = \frac{b/a}{\sin C} - \cot C.$$

For example, let $a = 225$ feet, $b = 350$ feet, $C = 48° 15'$. Here $a/b = 0.643$, $b/a = 1.556$, $\sin C = 0.746$, $\cot C = 0.893$; then $\cot B = -0.031$, $\cot A = 1.193$, whence $B = 91° 45'$ and $A = 40° 0'$. This is certainly close enough for blackboard work.

93

There are interesting relations between the sides of a triangle when the angles are related in value. Thus when angle $A$ is double the angle $B$ we have $a^2 = b^2 + bc$. When angle $A$ is three times the angle $B$, then we have $bc^2 = (a + b)(a - b)^2$.

94

In the American Mathematical Monthly for March, 1914, Artemas Martin finds many triangles with integral sides,

one angle being 60°. Thus the sides 8, 3, 7 give a triangle in which the angle opposite 7 is 60°; so also the sides 8, 5, 7 give such a triangle. Only two other triangles can here be mentioned: 15, 8, 13, and 15, 7, 13.

## 95

## ANALYTIC TRIGONOMETRY

The deductions of the formulas expressing relations between the functions of angles is often called analytic trigonometry. For instance $(2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta) = 2\cos 5\theta$ is a formula for the quintisection of an angle. Also $(2\cos\theta)^3 - 3(2\cos\theta) = 2\cos 3\theta$ is the formula for the trisection of an angle. When $5\theta$ or $3\theta$ is given $\cos\theta$ can be algebraically expressed, but such expressions cannot be handled numerically since they contain the square root of a negative quantity. Among the many interesting formulas of higher trigonometry the following deserve special consideration:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} +, \text{etc.} \qquad (1)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} +, \text{etc.} \qquad (2)$$

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} +, \text{etc.}$$

In these formulas the values of $\theta$ are to be taken in radians; thus, for an angle of 30° the value of $\theta$ is $1/6\,\pi$, and for an angle of $\theta°$ the value of $\theta$ is $\pi\theta°/180$. In the last formula $e$ denotes the number 2.71828 . . . or the base of the Naperian system of logarithms. The symbol (!) denotes the product of the natural numbers; thus: $4! = 1 \times 2 \times 3 \times 4 = 24$.

Let $i$ denote the imaginary $\sqrt{-1}$, and in the last formula change $\theta$ to $i\theta$; then it becomes

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} +, \text{etc.}$$

This may be written in the form

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} +\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} +\right)$$

or

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3}$$

Similarly, replacing $\theta$ by $-i\theta$ there is found

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{4}$$

Adding these two equations and also subtracting the second from the first, there results

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad i\sin\theta = \tfrac{1}{2}(e^{i\theta} - e^{-i\theta}), \tag{5}$$

which are remarkable expressions for the sine and cosine in terms of the imaginary $\sqrt{-1}$. These wonderful formulas are due to the great mathematician Euler. What do these formulas mean?

## 96

## COMPLEX QUANTITIES

Equation (3) forms a basis for the extensive branches of vector analysis and quaternions, for $\cos\theta + i\sin\theta$ is what is called a complex number, the general expression for which is $a + ib$. To define and understand $a + ib$ it is first necessary to understand $i$. By the rules of simple algebra it is found that:

$$i^2 = (\sqrt{-1})^2 = -1, \quad i^3 = (\sqrt{-1})^3 = -\sqrt{-1} = -i,$$
$$i^4 = (\sqrt{-1})^4 = +1, \quad i^5 = (\sqrt{-1})^5 = +\sqrt{-1} = +i.$$

Now the following graphic representation agrees with these

results. In Fig. 33 let a line be drawn from $O$ toward the right to represent $+1$ and one of equal length be drawn to the left to represent $-1$. Also let a line be drawn upward from $O$ to represent $+i$ and one downward to represent $-i$. Now let multiplication by $i$ indicate turning a line of unit length through 90 degrees about the axis $O$. Then $+1 \times i = +i$, or the line $+1$ has been turned into the position shown by $+i$ in the figure. Also $+i \times i = -1$ or the line $+i$ has been turned to the position $-1$; also $-1 \times i = -i$, and $-i \times i = +1$. Thus with this graphic representation we *see* at once that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$ and $i^5 = +i$.

Fig. 33

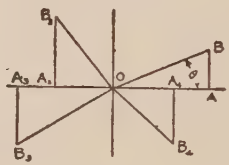



Fig. 34

Now to explain $a + ib$, let $OA$ in Fig. 34 be laid off to the right to represent $+a$, and then at $A$ lay off $AB$ at right angles to $OA$ to represent $+ib$. Then $OA + AB$ represents $a + ib$, or in other words $a + ib$ locates the point $B$. But the shortest way to go from $O$ to $B$ is by the hypothenuse $OB$, or by the vector addition $OA + AB = OB$. Here $OB$ may be called a vector and be indicated in general by $R$, so that $R = a + ib$ is the vector which locates a point $B$, this point being located either by going directly to it by the shortest distance $R$, or by stepping off $a$ units from $O$ toward the right and then $b$ units upward. When $a$ is negative and $b$ positive a point $B_2$ in the second quadrant

is located; when both $a$ and $b$ are negative a point $B_3$ in the third quadrant is located; when $a$ is positive and $b$ negative a point $B_4$ in the fourth quadrant is located.

The complex quantity $a + ib$ is frequently more conveniently expressed by

$$R = r(\cos\theta + i\sin\theta),$$

in which $r$ is the length of the vector $R$, while $r\cos\theta$ and $r\sin\theta$ represent $a$ and $b$. If this $R$ is squared it becomes

$$R^2 = r^2(\cos 2\theta + i\sin 2\theta),$$

and if it be raised to the $n$th power it becomes

$$R^n = r^n(\cos n\theta + i\sin n\theta),$$

which is known as the theorem of De Moivre. When $r = 1$ all the vectors are of unit length, and from the above formula (3) it is seen that

$$R = e^{i\theta} \quad \text{or} \quad e^{i\theta} = \cos\theta + i\sin\theta.$$

Fig. 35.

Hence $e^{i\theta}$ may be regarded as any radius in a circle of radius unit, this radius making an angle $\theta$ with the positive axis $OA$. The line $OB$ in Fig. 35 represents such a vector: if this be squared $OB$ revolves to the left through another angle $\theta$ and takes the position $OB_1$.

From the last equation several remarkable algebraic expressions may be derived:

| | |
|---|---|
| For $\theta = \frac{1}{2}\pi$, | $e^{\frac{1}{2}i\pi} = +i.$ |
| For $\theta = \pi$, | $e^{i\pi} = -1.$ |
| For $\theta = 3/2\,\pi$, | $e^{\frac{3}{2}i\pi} = -i.$ |
| For $\theta = 2\pi$, | $e^{2i\pi} = +1.$ |

These are wonderful expressions, for $e$ is the number 2.71828; algebraically or numerically they seem incomprehensible, but by the above graphic method of representation they are clearly understood.

## 97
## SPHERICAL TRIGONOMETRY

The Greek astronomers developed and used spherical trigonometry long before plane trigonometry was known. These scientists were led to study the spherical triangle because it was necessary in the solution of problems involving the altitudes, azimuths, and hour angles of the stars and planets. The first tables were those of the chords of the angles. Later the Hindoos introduced the sine instead of the chord. Then the Arabs further developed the theory of both plane and spherical triangles.

The sum of the angles of a spherical triangle is always greater than 180 degrees, and the excess over 180 degrees depends on the area of the triangle. A rough rule for the spherical triangles measured in geodetic surveys is that there is one second of spherical excess for each 76 square miles of area. The same rule applies to spherical polygons. Thus, a triangle or polygon of the size of the state of Connecticut has a spherical excess of about 64 seconds. A trirectangular triangle which covers one-eighth of the surface of the earth has a spherical excess of 90 degrees, for the sum of its angles is 270 degrees.

When two plane triangles are equal one can always be made to coincide with the other, either by motion along the plane or by turning it over on one of the sides as an axis. But there can be two spherical triangles which have their sides and angles equal each to each, and yet it is impossible by any kind of motions to bring them into coincidence. They must, however, be called equal, since every part of one is equal to a corresponding part in the other and their areas are the same. Here is a case where equal things cannot coincide or be imagined to coincide.

## 98
## HYPERBOLIC TRIGONOMETRY

Since 1875 there has been developed the interesting subject called Hyperbolic Trigonometry. This has nothing whatever to do with triangles, but it is intimately connected with a rectangular hyperbola. In the circle of Fig. 36 let $P$ be any point on the circle and from it let the perpendicular $PS$ be dropped upon the radius $OA$; also at $A$ let the perpendicular $AT$ be drawn until it meets the radius $OP$ produced. Let $AOP$ be the angle $\theta$, then if the radius $OA$ is unity, $OS$ is $\cos\theta$, $SP$ is $\sin\theta$, $AT$ is $\tan\theta$, and $OT$ is $\sec\theta$.

In the right-hand diagram of Fig. 36 let $OA$ be the semi-major axis of an equilateral hyperbola, and $P$ any point on

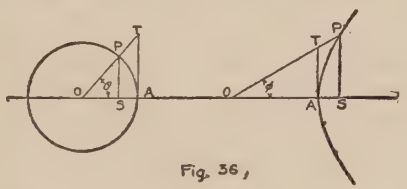



Fig. 36,

the curve. From $P$ drop $PS$ upon $OA$ produced, and from $A$ draw $AT$ perpendicular to $AO$ until it meets $OP$. Then if $AO$ is unity, $OS$ is called the hyperbolic cosine, $SP$ the hyperbolic sine, $AT$ the hyperbolic tangent, and $OT$ the hyperbolic secant. Let $\phi$ be double the area of the hyperbolic sector $OAP$, then

$$OS = \cosh\phi, \quad SP = \sinh\phi, \quad AT = \tanh\phi, \quad OT = \operatorname{sech}\phi,$$

where cosh means hyperbolic cosine, tanh means hyperbolic tangent, and so on. In academic slang these are often pronounced cosh, shin, than, shec.

In the circle $\overline{OP}^2$ equals $\overline{OS}^2 + \overline{SP}^2$, or $\cos^2\theta + \sin^2\theta = 1$. In the equilateral hyperbola $\overline{OP}^2$ equals $\overline{OS}^2$ minus $\overline{SP}^2$ or

$\cosh^2 \phi - \sinh^2 \phi = 1$. The letter $\theta$ denotes an angle or a multiple or submultiple of $\pi$. The letter $\phi$ denotes a number which may have any value.

The formulas (1) to (5) in No. 95 are true whether $\theta$ be real or imaginary. Changing $\theta$ to $i\theta$ in formula (5) and then replacing $\cos i\theta$ by $\cosh \theta$ and $-i \sin i\theta$ by $\sinh \theta$, they become

$$\cosh \theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta}), \quad \sinh \theta = \tfrac{1}{2}(e^{\theta} - e^{-\theta}), \qquad (6)$$

which are the values of the hyperbolic cosine and sine in exponential form. Here as always, the letter $e$ denotes the number 2.71728, the base of the hyperbolic system of logarithms. Let these expressions be squared and the second subtracted from the first, then $\cosh^2 \theta - \sinh^2 \theta = 1$, which agrees with the equation established in the preceding paragraph, $\theta$ being here the double of a hyperbolic sector.

The computation of the cosine and sine of an angle cannot be made by the formulas (5) since there is no way of obtaining the imaginary power $e^{i\theta}$. But the computation of the hyperbolic cosine and sine is easily made from (6); for instance let $\theta = 2$, then $e^2 = 7.389057$ and $e^{-2} = 0.135335$, hence $\cosh 2 = 3.626861$.

As the quantity $\theta$ varies from 0 to $\infty$, $\cosh \theta$ varies from 1 to $\infty$ and $\sinh \theta$ from 0 to $\infty$, while $\tanh \theta$ varies from 0 to 1. There is no periodic repetition of the real values of the hyperbolic functions as is the case with the circular functions, but yet they have imaginary periods.

Enough has now been stated to give the young reader a glimpse of the fundamental ideas at the foundation of hyperbolic functions. But, he may ask, of what use or importance are they? The reply to this is, that such functions constantly turn up in the solution of practical prob-

lems. For example, take the catenary, which is the curve assumed by a cord or cable suspended from two points and hanging freely under its own weight. Let $y$ be the ordinate or height of any point of the curve above a certain horizontal plane, $x$ the distance of the point to the right or left of the lowest point of the curve and $c$ a certain constant, then the equation of the curve is

$$y = \tfrac{1}{2} c \left(e^{x/c} + e^{-x/c}\right) = c \cosh x/c.$$

This equation in terms of $e$ was deduced a hundred years or more before hyperbolic cosines were ever thought of, but the second form in terms of cosh has many practical advantages over it.

Hyperbolic functions also turn up in the theory of arches, in the formula for a beam subject to both flexure and tension, in the construction of charts to represent large portions of the earth's surface, and especially in the electrical discussion of alternating currents. The length and areas of many curves which were formerly stated in terms of Naperian logarithms are now more conveniently expressed by hyperbolic functions.

# CHAPTER V

## ANALYTIC GEOMETRY

### 99

DESCARTES, a French philosopher, who lived in the first half of the seventeenth century, devised the method of coördinates by which curves can be graphically represented and their properties be studied through their equations. In this method, as applied to plane curves, two lines called axes are drawn at right angles. Their intersection $O$ is called the origin of coördinates. Values of $x$ are laid off parallel to the $X$-axis, and values of $y$ parallel to the $Y$-axis. Positive values of $x$ are laid off to the right of the $Y$-axis and negative ones to its left. Positive values of $y$ are laid off upward from the $X$-axis and negative ones downward. Thus if a point has the coördinates $x = 3$, $y = 2$, it is located at $A$; if it has the coördinates $x = -1$, $y = 1$, it is located at $B$; if it has $x = -2$, $y = -1$, it is at $C$; and if it has $x = 4$, $y = -3$, it is at $D$.

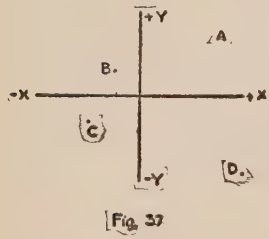


Fig. 37

### 100

The equation of a curve is an equation which gives the relation between the coördinates of any point on the curve. Thus $x/4 + y/3 = 1$ gives the relation between $x$ and $y$ for

every point; if $y = 3$ then $x = 0$, if $y = 6$ then $x = -4$, if $y = -3$ then $x = 8$. Plotting these three points by laying off their coördinates from the $X$- and $Y$-axes, it is seen that they are on one straight line. When $y = 0$ the line crosses the $X$-axis at $x = 4$; when $x = 0$ the line crosses the $Y$-axis at $y = 3$; and its inclination to the $X$-axis is the slope of 3 to 4. Thus any equation of the first degree between two variables is the equation of a straight line. This line is of infinite length for no matter how great $y$ is taken the corresponding value of $x$ can be found from the equation.

The equation of a circle is $x^2 + y^2 = r^2$ and that of an equilateral hyperbola is $x^2 - y^2 = r^2$. Equations of many other curves are known to the student who reads these pages. By discussion of these curves we learn their shape, we draw tangents to them at given points, we find where two curves intersect, and later by the help of the calculus we can find their lengths and also determine the areas included between them and the axes.

## 101

Coördinates are used also for the graphic representation of statistics and natural phenomena. For example, let the following be the means of the mean monthly temperatures at a certain place for a series of years:

| Jan. | Feb. | Mar. | Apr. | May | June | July | Aug. | Sept. | Oct. | Nov. | Dec. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20° | 28° | 28° | 34° | 47° | 60° | 78° | 77° | 66° | 47° | 38° | 25° |

If the reader will place twelve points at equal distances apart on the $X$-axis, and then lay off upward the temperatures given, 20 for Jan. at the first point, 28 for Feb. at the second point, and so on, and then connect these points with a curve, he will have a graphic representation of the mean

monthly temperatures throughout the year at the given place. This curve resembles the curve of sines, that is the curve whose equation is $y = \sin x$. In plotting the points through which the curve is to be drawn, it is best to use paper that is ruled into squares.

A point to be plotted is sometimes indicated by the notation (5, 3) where the first number is the coördinate $x$ and the second the coördinate $y$. Let the reader plot the following ten points, numbering them 1, 2, 3, etc., and then join the points by a curve in the order of the numbers: (1.2, 2.0), (2.0, 1.1), (3.0, 1.0), (4.0, 1.7), (4.1, 2.6), (3.0, 3.1), (2.0, 3.5), (2.1, 4.6), (3.0, 5.0), (3.8, 4.6).

## 102

Let it be required to find the intersection of the circle $x^2 + y^2 = 16$ with the straight line $3x + 5y = 15$. By roughly plotting the circle and the line there will be found the two points (3.9, 0.6) and (−1.4, 3.8). But closer values can be found by finding the values of $x$ and $y$ by combining the two equations; this gives 3.95 and −1.31 for the two values of $x$, and 0.63 and 3.79 for the corresponding values of $y$.

Now let it be required to find the intersection of the circle $x^2 + y^2 = 16$ with the straight line $x + y = 8$. Plotting the circle and the line it is seen that they do not intersect. But combining the two equations there are found for $x$ the two values $4 + \sqrt{-8}$ and $4 - \sqrt{-8}$, and for $y$ the corresponding values $4 - \sqrt{-8}$ and $4 + \sqrt{-8}$. These imaginary values show, of course, that the straight line does not intersect the circle. But have they no other meaning? Yes, they have a definite geometric meaning which will be explained later.

Let it be required to plot the curve whose equation is

$$\pm x = \tfrac{1}{2} y - a\sqrt{1 - \left(\frac{y-a}{a}\right)^{80}}.$$

Assume values of $y$ and find the corresponding values of $x$; thus, when $y$ is negative then $x$ is imaginary, when $y = 0$ then $x$ is 0, when $y$ is very small then $x = \pm a$, when $y = \frac{1}{2} a$ then $x = \pm \frac{3}{4} a$, when $y = a$ then $x = \pm \frac{1}{2} a$, when $y = 3/2\, a$ then $x = \pm \frac{1}{4} a$, as $y$ approaches $2\, a$ then $x$ approaches 0, when $y = 2\, a$ then $x = \pm a$, when $y$ is greater than $2\, a$ then $x$ is imaginary. Accordingly, Fig. 38 represents the real curve. The key-note to the formation of this equation lies in the quantity $(y - a)/a$ which is raised to the 80th or any other large even power. When this quantity is numerically less than unity its 80th power is very small; for example, let $y = a/10$, then the fraction is $-0.9$ and its 80th power is 0.00021. This subtracted from 1 gives 0.99979 the square root of which is almost 1. Thus on any practicable scale of plotting, Fig. 38 is a proper representation of the equation.

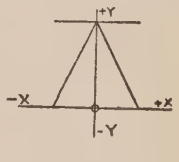


Fig. 38

## 103

### TRANSCENDENTAL CURVES

Curves involving circular functions are often called transcendental. Like the sine curve, $y = \sin x$, they have a period $2\, \pi$ and hence repeat themselves in both directions to infinity. For example, the curve whose equation is $\sin^2 y = \sin x \sin \frac{1}{4} x$ consists of the series of points, ovals, and lemniscates shown in Fig. 39. Here both $x$ and $y$ are taken in radians. When $x = \pi$ or $2\, \pi$ then $\sin^2 y = 0$,

$\sin y = 0$, or $y = 0, \pi, 2\pi$, etc., and this gives the vertical column of points. When $y = \pi$ or $2\pi$ then either $\sin x = 0$ or $\sin \frac{1}{4} x = 0$, whence $x = \frac{1}{2}\pi, \frac{3}{2}\pi$, etc., or $2\pi, 6\pi$, etc. When $\sin x \sin \frac{1}{4} x$ is negative then $y$ is imaginary. Thus with much labor the curves in Fig. 39 are constructed.

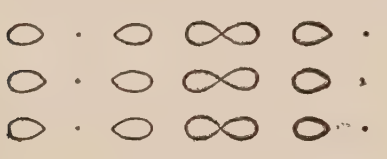


Fig. 39

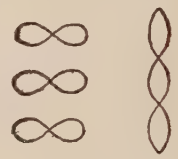


Fig. 40

When the equation is $\sin^2 y = \sin x \sin 1/5\, x$ some parts of the diagram undergo great changes and Fig. 40 results. Many beautiful diagrams of such curves, constructed by Newton and Phillips, may be seen in the Transactions of the Connecticut Academy for 1875.

The curve $\tan^2 x + \tan^2 y = 100$ gives a series of squares spaced like the plan of a rectangular city. When $x$ and $y$ are taken in degrees, each side of a square is 168° 36′ and the width of the streets between the squares is 11° 24′.

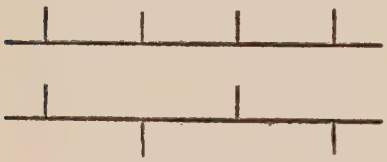


Fig. 41

The equation $y = \sin^{80} x$ gives a diagram like the upper one in Fig. 41 while the equation $y = \sin^{81} x$ gives the lower one. Both of these extend right and left to infinity.

The equation $\sin^2 \sqrt{x^2 + y^2} = 0$ is satisfied only when $\sqrt{x^2 + y^2} = 0°, 180°, 360°, 540°$, etc. When $\sqrt{x^2 + y^2} = 180°$ then $x = 180°$ and $y = 144°$ or $x = 144°$ and $y = 108°$.

When $\sqrt{x^2 + y^2} = 360°$ then $x = 216°$ and $y = 288°$ or $x = 288°$ and $y = 216°$. Hence the equation represents a series of points radiating from the origin, each point enjoying the property that its $x$, $y$, and $r$ form the sides of a right-angled triangle which are in the ratio $3 : 4 : 5$, or in the ratio $4 : 3 : 5$.

The equation $y^2 = \sin x$ represents a series of circles in a horizontal row.

The equation $y^2 = x^2$ represents two intersecting straight lines which are inclined at angles of 45 to the rectangular axes. But the equation $y^2 = x^2 \sin x$ represents a series of ovals included between such lines as the envelope. Also the equation $y^2 = ax$ represents the common parabola, but the equation $y^2 = ax \sin x$ represents a series of ovals included within the parabola as an envelope.

## 104

### POLAR COÖRDINATES

When a point is located by its distance from an origin $O$ and the angle $\theta$ which that distance, or radius vector, makes with the horizontal $X$-axis, the case is one of polar coördinates. Any equation in rectangular coördinates may be transformed into one in polar coördinates by replacing $x$ by $R \cos \theta$ and $y$ by $R \sin \theta$, where $R$ is the radius vector. Thus the parabola $y^2 = ax$ is expressed in polar coördinates by $R = a/\tan \theta \sin \theta$.

## 105

The equation $R = 10 + 10 \cos \theta$ gives an interesting curve called the cardioid. When $\theta = 0°$ then $R = 20$, when $\theta = 60°$ then $R = 15$, when $\theta = 90°$ then $R = 10$, when $\theta = 120°$ then $R = 5$, when $\theta = 180°$ then $R = 0$.

This gives the upper half of the curve and the lower half is similar to it. This curve is entirely appropriate for a young man to send to a young woman in the month of February as a mathematical valentine, provided she understands polar coördinates.

The equation $R = 10 + 20 \cos \theta$ is called the trisectrix, because, after the curve is drawn it may be used for the trisection of an angle. For explanation see the Encyclopedia Britannica.

The equation $R = 5 \pm \sqrt{\sin 20\,\theta}$ represents twenty small circles arranged around a central point. When $\theta = 0°$ then $R = 5$, when $\theta = 4.5°$ then $R = 5.7$ or 4.3, when $\theta = 9°$ then $R = 5$, when $\theta$ is between $9°$ and $18°$ then $R$ is imaginary; when $\theta = 18°$ then $R = 5$ and another small circle begins.

The equation $R = 10 \pm \sin \theta$ represents another curious curve, the discussion of which will be left to the ingenuity of the reader.

## 106

## REMARKS

Paradoxes and fallacies decrease as we ascend the mathematical ladder. Under Arithmetic we had a dozen or two, under Algebra nearly as many, under Geometry a few, under Trigonometry only one. In Analytic Geometry H. E. Licks tried hard to find a fallacy, but without success.

The curves formed by the intersection of a plane with a right cone are called conic sections. If twenty persons, not mathematicians, are asked what curve is formed when the plane is parallel to the base of the cone, they will reply that it is a circle. If they are asked what is the curve when the plane is slightly inclined to the base, they will reply that

it is an oval with the larger end nearest the base; this answer is also generally given by those who have studied Analytic Geometry and have rarely thought of the subject since their college days. Yet this curve is an ellipse which is no blunter at the lower end than at the upper, and Analytic Geometry furnishes a rigorous proof of it. The curve is always a true ellipse until the plane has the same inclination as the side of the cone, then it is a parabola. For greater inclinations the curve is a hyperbola. Hence, the circle, ellipse, parabola, and hyperbola are called the conic sections.

Analytic Geometry does not furnish an array of interesting facts. To draw a tangent to a curve at a given point is not an interesting exercise, nor is the determination of the angle included between two lines. The important things in Analytic Geometry seem to be, (1) the plotting of curves from given coördinates, (2) the plotting of curves from their equations, and (3) the construction of an equation to represent a series of plotted points. None of these receive proper attention in the text-books; indeed, the last is rarely or never alluded to.

Formal rules for solving certain problems are liked by teachers, but in reality they introduce a repression of reasoning which is harmful to the student. For example, in the American Mathematical Monthly for April, 1914, there is given the following as the "usual rule" for drawing a line through a given point parallel to a given line. "Step 1: Change the constant term in the given equation to $k$. Step 2: Substitute the coördinates of the given point in the equation of step 1 and solve for $k$. Step 3: Substitute this value of $k$ in the equation of step 1, and the result is the desired equation." Now what real knowledge of Analytic Geometry has the student gained by this procedure? He

has gained discipline in obeying orders but no development in his reasoning powers has resulted. If this is the way the subject is taught, it is indeed a case of the blind leading the blind.

On each blackboard of a school where this subject is taught there should be in one corner a permanent ruling of about a hundred squares each one inch in size. When the problem is given to draw a line through a point parallel to a given line, the student should put rectangular axes on this network of squares, then plot the given point, next draw the given line by finding the points where it intersects the axes, then draw freehand a parallel line through the given point. He will then see that the two lines have the same slope, and it should be left to his own ingenuity to find an equation to represent the second line. Two exercises like this do more to develop the reasoning powers of the student than a hundred exercises solved by formal rules with "steps."

## 107

## GRAPHIC SOLUTION OF EQUATIONS

Let it be required to solve the cubic equation $x^3 + ax + b = 0$. This equation can be supposed to result from the elimination of $y$ from the two simultaneous equations $y = x^3$ and $y = -ax - b$. The curve whose equation is $y = x^3$ can be plotted on squared paper by the help of a table of cubes, then if the straight line whose equation is $y = -ax - b$ is drawn on this plot the abscissas of its points of intersection with the curve are the roots of the equation $x^3 + ax + b = 0$. When the line intersects the curve in only one point, the equation has one real root and two imaginary roots; when it intersects the curve in three points there are three real roots. Thus let $x^3 - 20x +$

$30 = 0$ be the given equation. In Fig. 42 the full line represents the equation $y = x^3$ and the broken line the equation $y = 20x - 30$, the horizontal scale being twenty times the vertical scale. Here there are three real roots $x_1 = +1.8$, $x_2 = +3.3$, $x_3 = -5.1$.

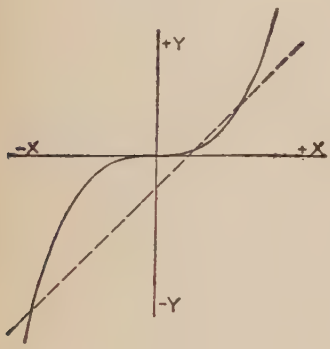



Fig. 42

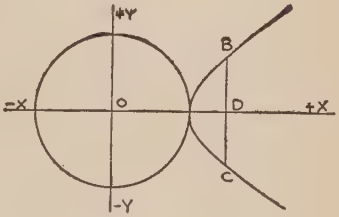



Fig. 43

The cubic equation $y^3 + Ay^2 + Bx + C = 0$ can be reduced to the form $x^3 + x^2 + bx + c$ by putting $y = A^{\frac{1}{3}}x$ and then dividing by $A$. Here the curve whose equation is $y = x^3 + x^2$ can be plotted by the help of tables of cubes and squares, then the points where the straight line $y = -ax - b$ intersects the curve give the roots of the cubic equation.

Let it be required to find the points where the straight line $x = 5$ intersects the circle $x^2 + y^2 = 16$. The algebraic solution gives $y = \pm 3\sqrt{-1}$ and these values may be represented graphically in the following manner. Let the hyperbola $x^2 - y^2 = 16$ be drawn, with its vertex at $A$, in a plane at right angles to that of the axes $XX$ and $YY$. Let $OD = x = 5$, then $DB$ and $DC$ are each equal to 3 with

respect to the hyperbola but each is equal to $3\sqrt{-1}$ with respect to the circle. Many interesting illustrations of this method of representing the imaginary and complex roots of equations may be seen in Graphic Algebra by Phillips and Beebe.

# CHAPTER VI

## CALCULUS

### 108

SIR ISAAC NEWTON'S method of obtaining the differentials of quantities is given in Section II, Book II, Lemma II of his great Principia. It is remarkable that in this discussion there is no reference to infinitesimal quantities, or to the vanishing of their powers and products, or to the doctrine of limits. The discussion is clear and simple, and well adapted to the comprehension of a beginner in calculus. Hence space may well be taken here to explain his method. The reader may, perhaps, most conveniently refer to the American edition of Newton's Principia, or Principles of Natural Philosophy, published at New York in 1848, where the discussion begins on page 262.

Newton supposes the sides $A$ and $B$ of a rectangle to vary from smaller to larger values. When the sides had the values $A - \frac{1}{2}a$ and $B - \frac{1}{2}b$, its area was $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$. When the sides are increased to $A + \frac{1}{2}a$ and $B + \frac{1}{2}b$, the area becomes $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From this latter rectangle subtract the former and there remains $aB + bA$. "Therefore," says Newton, "with the increments $a$ and $b$ of the sides, the increment $aB + bA$ of the rectangle is generated." In the modern notation $x$ and $y$ are taken as the variable sides of the rectangle and $dx$ and $dy$ as their increments or differentials, then in the above dis-

cussion replacing $A$ by $x$ and $B$ by $y$, also $a$ by $dx$ and $b$ by $dy$, there is found $x \cdot dy + y \cdot dx$ as the differential or increment of the area $xy$. Thus $dxy = x \cdot dy + y \cdot dx$ is deduced without considering $dx$ and $dy$ to be infinitely small. In the following derivation of differentials modern notation will be used instead of that employed by Newton.

Numerous algebraic functions can be differentiated by use of the formula $dxy = x \cdot dy + y \cdot dx$. To find the differential of $x^2$ make $y = x$ and $dy = dx$; then $dx^2 = 2\,x\,dx$. To find the differential of $x^3$, make $y = x^2$ and $dy = 2\,x\,dx$ in the formula; then $dx^3 = 3\,x^2\,dx$. To find the differential of $x^4$, make $y = x^3$ and $dy = 3\,x^2\,dx$ in the formula; then $dx^4 = 4\,x^3\,dx$. From these, by induction, it is concluded that $dx^n = nx^{n-1}\,dx$, and, as in the case of the binomial formula, we infer correctly that this applies whether $n$ be a whole number or a fraction, whether it be positive or negative. Thus $dx^{-1} = -x^{-2}\,dx$, and $dx^{n/m} = (n/m)x^{n/m-1}\,dx$.

Also in the formula $d\,xy = x \cdot dy + y \cdot dx$, the letters $x$ and $y$ may indicate any variable quantities. Thus to find the differential of the fraction $u/v$ let $x = 1/v, dx = -dv/v^2$, $y = u$, $dy = du$ in the formula, then $du/v = (1/v)\,du - (u/v^2)\,dv$. This may be written in the form $(v\,du - u\,dv)/v^2$, that is, the differential of a fraction equals its denominator into the differential of its numerator minus its numerator into the differential of its denominator, divided by the square of the denominator.

## MAXIMA AND MINIMA

### 109

The differential calculus enables easy the solution of many problems involving maxima and minima For example, a tin cylindrical box of diameter $a$ and height $h$ is to

be made to contain $Q$ cubic inches of material. If the thickness of the tin is $t$ what must be the ratio of the height to diameter in order that the least amount of tin may be used? Here the quantity of tin is $(\pi ah + \frac{1}{2}\pi a^2)\, t$, this including the cover of the box; also $Q = \frac{1}{4}\pi a^2 h$. Taking the value of $h$ from the second equation and substituting it in the first gives $(Q/a + \frac{1}{2}\pi a^2)\, t$ as the expression for the quantity to be made a minimum. Placing the derivative of this equal to zero and solving, there results $a^3 = 4\,Q/\pi$ or $Q = \frac{1}{4}\pi a^3$ and equating this to the above value of $Q$, there is found $h = a$. Hence, for minimum material the height of the box must be equal to its diameter.

## 110

As a second example, let it be required to find the length of the longest straight stick $AB$ which can be put up a circular shaft in the ceiling of a room, the height of the room being $h$ and the diameter of the shaft $a$. Here it is convenient to let $\theta$ be the angle which the stick makes with the floor; then $AB = h/\sin\theta + a/\cos\theta$. Placing the derivative equal to zero, there results $\tan\theta = (h/a)^{\frac{1}{3}}$. Then expressing $\sin\theta$ and $\cos\theta$ in terms of $(h/a)^{\frac{1}{3}}$ there is found for the length of the stick $AB = (h^{\frac{2}{3}} + a^{\frac{2}{3}})^{\frac{3}{2}}$. This is a simple way to solve a problem which has proved a stumbling block to many.

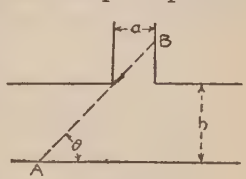


Fig. 44

## 111

Hundreds of problems similar to the above may be found in books and mathematical journals, hence H. E. Licks gives one not found in books, namely, to determine the path

of a ray of light from a source $S$ to the eye at $E$, when a transparent glass plate is interposed between them. Let

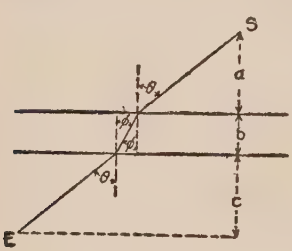


Fig. 45

Fig. 45 show the path by the heavy broken line, the light moving in straight lines both within and without the plate, as is known by experiment. Let $a$, $b$, $c$, $d$ be the distances between $S$ and $E$ measured normal and parallel to the plate. Let $\theta$ be the angle which the ray makes with the normal to the plate before it enters and after it leaves, and $\phi$ the angle which it makes with that normal within the plate. Let $v_1$ be the velocity of the light without the plate and $v_2$ the velocity within it. Then the time required to travel from $S$ to $E$ is

$$t = a \sec\theta/v_1 + b \sec\phi/v_2 + c \sec\theta/v_1.$$

Also the quantities are connected by the geometric relation $a \tan\theta + b \tan\phi + c \tan\theta = d$. Now the path must be such as to make the time $t$ a minimum. Hence, if $N$ is a constant to be determined, the quantity

$$t = (a+c) \sec\theta/v_1 + b \sec\phi/v_2 + N[(a+c) \tan\theta + b \tan\phi - d]$$

is to be made a minimum. Differentiating there is found

$$\frac{dt}{d\theta} = (a+c) \sec\theta \tan\theta/v_1 + N(a+c)/\cos^2\theta = 0,$$

$$\frac{dt}{d\phi} = b \sec\phi \tan\phi/v_2 + Nb/\cos^2\phi = 0.$$

From the first of these $N + \sin\theta/v_1 = 0$ and from the second $N + \sin\phi/v_2 = 0$, whence by elimination of $N$, there is found $\sin\theta/\sin\phi = v_1/v_2$. Hence the ratio of the sines of the angles made by the ray with the normal of the plate is

equal to the ratio of the velocities of light without and within the plate. Thus the path is completely determined in terms of the velocities $v_1$ and $v$.

The ratio of $\sin\theta/\sin\phi$ is, in optics, called the index of refraction and its values have been accurately determined by measurements for different materials. Thus when light passes from air into water this index is 1.33, that is, the velocity of light in water is about three-fourths of its velocity in air.

## 112

### THE CELL OF THE HONEY BEE

The cell made by the bee in which to store honey is shown in Fig. 46. The end *ABDE* is the top of the cell which is closed with a plane cap after the cell is filled with honey. The cross-section of the cell is a regular hexagon formed with thin sides of wax. The bottom of the cell *abdefg* is terminated by three equal planes which meet at the apex *c* and which are rhomboidal in shape so as to form a depressed cup, for the points *a*, *f*, *d* are further away from the top of the cell than are the points *b*, *e*, *g*, and the apex *c* is still further away. The angles of these rhomboids at *b*, *e*, *g* are equal to the angles at *c*. If a cross-section of the cell be taken anywhere on its length, there results a hexagon each of whose interior angles is 120°, but the six angles in the bottom of the cell at *b*, *e*, *g*, *c* are only about 110° owing to the inclination of the three planes.

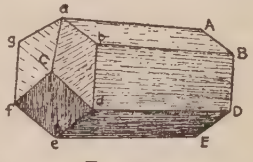


Fig. 46

It is evident that there is a certain inclination of these planes which will give less material for the cell than if the lower end were made plane like the upper end. To deter-

mine this inclination is a problem in maxima and minima which has received much attention because the conclusion deduced agrees closely with the actual construction of the cell. Fig. 47 gives end and side views of the cell. Let $h$ be the mean length of the cell, and $h - x$ the length of the side $Bb$. Regarding $abdefg$ as the cross-section let each of its sides be called $r$, then $bc$ is also $r$; but by virtue of the inclination of the rhombus $abdc$ the distance $bc$ becomes increased as shown in Fig. 48. Let this increased distance be called $t$. The plane $abdc$ in Fig. 47 then has an inclination such that the distance $t$ is $\sqrt{r^2 + 4x^2}$.

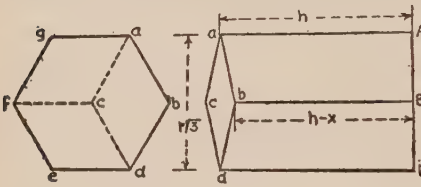



Fig. 47

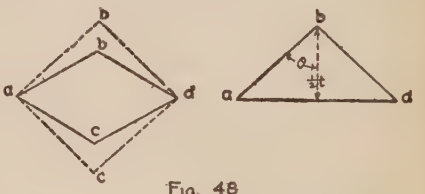



Fig. 48

Now considering the amount of wax in the cell to be proportional to the sum of the areas of its sides and bottom, the expression for the total area in terms of $x$ is made a minimum. The area of the two sides shown in the side view is $r(2h - x)$, the area of the inclined rhombus $abdc$ is $\frac{1}{2} t \times r\sqrt{3}$ or $\frac{1}{2} r\sqrt{3}\sqrt{r^2 + 4x^2}$, and hence the total area $A$ to be made a minimum is

$$A = 3\left[r(2h - x) + \tfrac{1}{2} r\sqrt{3r^2 + 12x^2}\right].$$

Differentiating this value of $A$ with respect to $x$, equating the derivative to zero, and solving, gives $x = r\sqrt{1/8}$ for the

value of $x$ which renders the quantity $A$ a minimum. For this value of $x$ the area $A$ becomes $A_1 = 3r(2h + \frac{1}{2}r\sqrt{2})$ which is proportional to the amount of wax in the sides and bottom of one cell.

If the cell had a plane bottom at right angles to the sides, the area of the sides and bottom is found by making $x = 0$ in the above expression for $A$, whence $A_0 = 3r(2h + \frac{1}{2}r\sqrt{3})$.

The ratio of $A_0$ to $A_1$ now is

$$s = \frac{2h + \frac{1}{2}r\sqrt{3}}{2h + \frac{1}{2}r\sqrt{2}} = \frac{4h/r + \sqrt{3}}{4h/r + \sqrt{2}}$$

and the following are values of this ratio for various values of $h/r$:

| | |
|---|---|
| For $h/r = 0$, | $s = 1.225$. |
| For $h/r = 1$, | $s = 1.072$. |
| For $h/r = 2$, | $s = 1.034$. |
| For $h/r = 4$, | $s = 1.018$. |
| For $h/r = 6$, | $s = 1.013$. |

It hence appears that the saving in wax of the actual cell over a cell with plane bottom is 7.2 per cent when $h = r$, 3.4 per cent when $h + 2r$, and 1.8 per cent when $h = 4r$. The height of the cell is usually between $h = 2r$ and $h = 5r$, so that the saving in wax is on the average about 2 per cent.

Early writers on this problem paid great attention to the angles *abd* and *acd* of the rhombus in Fig. 47. The tangent of the angle $\theta$ (Fig. 48) when $x$ has the value $r\sqrt{1/8}$ which renders $A$ a minimum, is $\frac{1}{2}r\sqrt{3}/\frac{1}{2}t$, and since $t = r\sqrt{3/2}$ this tangent is $\sqrt{2}$. Then by the help of a logarithmic table it is easy to find $\theta$, and its double 109° 28′ 16″ is the angle *abd* in the inclined rhombus, and this is also the value of each of the angles at the apex $c$ of the pyramidal cup.

Statements were made that this angle had been measured and found to be 109° 28′, from which it was concluded that the cell of the bee agreed most closely with that which theory demanded for the minimum quantity of wax. However, evidence regarding these measurements is wanting, and indeed it would be a very difficult matter to measure this angle to such a degree of exactness.

This problem first received discussion in the eighteenth century, and writers on it generally extolled the wonderful instinct of the bee in adopting a form of cell which led to economy in wax. The production of wax is an exhausting

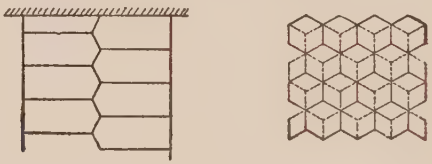


Fig. 49

operation for the bee, and moreover sixteen pounds of honey are needed to produce one pound of wax. Economy is hence promoted by any method which will limit the production of wax to the least possible amount. According to most writers the bee has solved this problem in a most ingenious mathematical manner, and its instinct should be regarded as one of the most remarkable in nature.

In order to judge how far these high enconiums are justified, it is necessary to examine the construction of the honeycomb. An inspection of one shows that it is formed by two tiers of horizontal cells with their bases resting on a vertical midrib in which the pyramidal cups are formed. In Fig. 49 the heavy lines of the right-hand diagram give a front view of the cells on one side of the midrib, and the broken lines

show the cells on the other side. These two tiers of cells alternate in a curious manner, the bottom of one cell abutting against the bottoms of three cells of the other tier. The cells themselves are either horizontal or inclined very slightly upward, and the left-hand diagram in Fig. 49 shows a vertical section before they are filled with honey. Both tiers of the comb are supported by the central midrib which is attached to the ceiling of the hive. In building the cells the bees begin at the top and work downward, the base of each cell being of course built before the cell itself.

The examination of such a honeycomb will also show that the midrib forming the bases of the cells is thicker than the walls of the cell itself, this probably being so because it is required to carry all the weight of the cells and honey. In fact it has been stated that the midrib is thicker near its top than lower down. The observations of the writer lead to the rough conclusion that the midrib is $1\frac{1}{2}$ or 2 times thicker than the walls of the cells. This being the case, the above theory falls to the ground as fallacious.

Let $n$ be the ratio of the thickness of the midrib to that of the walls of the cell. Then the above expression for the area $A$ becomes

$$A = 3r\left(2h - x + \tfrac{1}{2}n\sqrt{3r^2 + 12x^2}\right).$$

The value of $x$ which renders this a minimum is now found to be $x = \frac{1}{2}r/\sqrt{3n^2 - 1}$. From this the following values are found for the angle $abd$ of the inclined rhombus and for the angles at the apex $c$:

| | |
|---|---|
| For $n = 1$, | $abd = 109^\circ\ 28'\ 16''$. |
| For $n = 1\frac{1}{2}$, | $abd = 116^\circ\ 40'\ 0''$. |
| For $n = 2$, | $abd = 117^\circ\ 59'\ 10''$. |
| For $n = 4$, | $abd = 119^\circ\ 38'\ 58''$. |

Here the last value is given in order to show that the bottom of the cell becomes practically flat when the midrib is four times as thick as the walls of the cell. For a perfectly flat bottom this angle is of course exactly 120°.

While this variation in thickness of the midrib appears to take the problem outside of the domain of pure mathematics, yet such is not really the case. Exact observations of the way in which the bees build the comb are needed, as also measurements of the bases of the cells, and perhaps these may be made in the laboratories of natural history. At present the writer offers the following as conjectures: (1) that the cell of the bee is built according to the rules deduced above for minimum material when the midrib is equal in thickness to the walls of the cell, (2) that this shape of the cell is not due to an instinct for securing the minimum quantity of wax, but is entirely due to a method of construction which arises from a necessity that the bees in adjoining cells should crowd together as closely as possible.

The first conjecture can only be established by measurements made on the same midrib at both upper and lower parts of the comb, and on different midribs in different kinds of cells. If the angles of the inclined faces of the apex $c$ of the pyramidal cup can be measured, the writer predicts that these will approximate to the value 109° 28′ 16″.

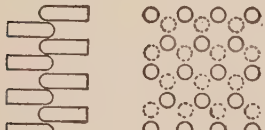


Fig. 50

Concerning the second conjecture it should be noted that the midrib is built by bees which face each other in the work as shown in the left-hand diagram of Fig. 50. In order that the midrib between the two tiers of cells may be properly compacted it is necessary that the heads of the

bees in one tier should alternate with the heads of those in the other tier. In the right-hand diagram the full-line circles show the heads of the bees in one tier and the broken-line circles the heads of those in the other tier. Here it is seen that each bee occupies a triangular position between three other bees, and with this arrangement it is indispensably necessary that the bottom of each cell should be a pyramidal cup having three sides.

In the theoretic cell deduced at the beginning of this article the inclination of each of the three planes of the bottom of the cell to a cross-section is such that the tangent of the angle is $2\,x/r$ or $\sqrt{\frac{1}{2}}$. This corresponds to $35^\circ\ 45'\ 52''$. The first conjecture of the author demands that this inclination should always be $35^\circ\ 45'\ 52''$ whatever be the thickness of the midrib. Further investigation of these three planes will show that the diedral angle between any two is exactly $120^\circ$, and this in the conjecture of the writer is always closely the case.

Let the reader take four spheres of equal size, lay three of them on a table so that each is tangent to the other two, and then put the fourth sphere upon these three. If these points be located and three tangent planes be drawn, it is a simple matter of computation to find that each plane makes an angle of $35^\circ\ 45'\ 52''$ with the horizontal, and that the diedral angle included between any two of the planes is $120^\circ$. Thus a series of alternating spheres gives the same planes as are found in the cells of the bee; hence one cause of the inclination of the latter is undoubtedly the alternating heads of the bees in forming the midrib shown in Fig. 50.

The reason why the cells are hexagonal has often been discussed. All writers are in agreement that this is due to

the circumstance that each cell is surrounded by six others, and that if any other form than the hexagonal were adopted vacant spaces would be left, which could not be filled with honey. Moreover it is thought by the writer that any cell wall must be built by bees working upon both sides of it. Now in the hexagonal cell the diedral angle between any two adjacent side walls is 120°. At the base of this cell, as we have seen, the diedral angle between any two of the planes forming the pyramidal cup is 120°; also each of these planes in intersecting a side of the hexagonal cell makes with it an angle of 120°. Hence in the bee cell every diedral angle is 120°. The angles at the top of the cell, where the cap is put on, are not here included; but as long as the bee is in the cell she has only to deal with diedral angles of 120°.

The conclusion of this discussion is that the cells of the bee are not built from any instinct for reducing the production of wax to a minimum, but rather from the necessity that their heads must alternate in forming the midrib in order to properly compact it. This necessity results in planes inclined to each other at angles of 120°. Perhaps it may be said that the bee has an instinct to build planes inclined at this angle, but more properly, it seems to the writer, it may be said that the work of the bees is more easily done in this way than in any other. Economy in labor rather than in material appears to lie at the foundation of the symmetric form of the cell of the industrious honey bee.

An interesting critical article by Glaisher will be found in the London Philosophical Magazine for August, 1873, where the history of this famous problem is set forth in full detail. At that date the belief appears to be undoubted that the form of the cell is due to an instinct of the bees for saving as

much wax as possible, and this is referred to as one of the most remarkable instances of instinct in nature. Since the discussion here given indicates otherwise, further investigations are in order to fully solve the problem, and these are only possible after many observations and measurements have been made in entomological laboratories.

## 113

## INTEGRAL CALCULUS

Differentiation is a definite process and any given function of a single variable can be differentiated. But there is no way to integrate except from a knowledge of what has been done in differentiation. In this respect the two branches of calculus are analogous to involution and evolution in arithmetic; any given number may be raised to a stated power, but when the power is given there is no way to find the root except by guess work and trial. There are about twenty-five fundamental integrals which are known to be correct because the differentiation of them furnishes the given expression with which we start. All the rest of integral calculus consists in reducing the quantity to be integrated to one of the fundamental forms.

For instance, $\int x^{n-1}\,dx$ equals $x^n/n$ because the differential of the latter is $x^{n-1}\,dx$ and for no other reason. Similarly $\int \sin x \cdot dx$ equals $-\cos x$ because the differential of $\cos x$ is $-\sin x \cdot dx$. In all cases the correctness of an integral is to be determined by differentiating it.

## 114

To the above statement that any function can be differentiated there seems one exception. Weirstrass has devised

a certain series, expressed in symbolic form, for which a derivative cannot be obtained, because in any interval, no matter how small, there are an infinite number of bends of the curve, so that at any given point it is not possible to draw a tangent to the curve. This expression, however, is little more than a curiosity to a beginner.

## 115

When $y \cdot dx$ is required, $y$ being expressed in positive integral powers of $x$, then the integral can be directly found from the formula

$$\int y \cdot dx = yx - D_1\frac{x^2}{2!} + D_2\frac{x^3}{3!} - D_3\frac{x^4}{4!} + D_4\frac{x^5}{5!} -, \text{etc.},$$

in which $D_1$, $D_2$, $D_3$, are the first, second, and third derivatives of $y$ with respect to $x$. For example, let $y = ax^2 + x^3$, then $D_1 = 2\,ax + 3\,x^2$, $D_2 = 2\,a + 6\,x$, $D_3 = 6$, $D_4 = 0$. Then, substituting in the formula, there is found

$$\int (ax^2 + x^3)\,dx = 1/3\,ax^3 + 1/4\,x^4.$$

Unfortunately this formula does not seem to apply to other functions which have no $D$ equal to 0 but in each given case we are forced to consult a catalog of integrals, or to reduce the given function to one whose integral is known.

## 116

John Phoenix, the first real humorist of America, was a graduate of West Point and hence well versed in mathematics. In his essay called "Report of a Scientific Lecture," he alludes to the importance of adding a constant to the result of an integration. He says:

By a beautiful application of the differential theory the singular fact is demonstrated, that all integrals assume the forms of the atoms of which they

are composed, with, however, in every case the important addition of *a constant*, which like the tail of a tadpole, may be dropped on certain occasions when it becomes troublesome. Hence, it will evidently follow that space is round, though, viewing it from various positions, the presence of the cumbersome addendum may slightly modify the definity of the rotundity. To ascertain and fix the conditions under which, in the definite consideration of the indefinite immensity, the infinitesimal incertitudes, which, homogeneously aggregated, compose the idea of space, admit of the computible retention of this constant, would form a beautiful and healthy recreation for the inquiring mind; but, pertaining more properly to the metaphysician than to the ethical student, it cannot enter into the present discussion.

## 117

## LENGTHS OF CURVES

The lengths of nearly all curves are expressed in terms of circular, hyperbolic, or logarithmic functions. Thus, the length of an arc of a circle is always in terms of $\pi$, and the length of an arc of a parabola is in terms of a hyperbolic logarithm. The story is told that a German professor, lecturing to his class two hundred years ago, said that the length of no curve could be algebraically expressed, and that the next day one of the students brought to him the derivation of the length of an arc of the semi-cubical parabola in algebraic terms. This curve has the equation $n^{\frac{1}{2}}y = x^{\frac{3}{2}}$. The derivative $dy/dx$ is $3/2\, n^{-\frac{1}{2}}x^{\frac{1}{2}}$ and the length of an arc between the limits $x = 0$ and $x = a$ is:

$$\int_0^a dx\sqrt{\frac{9x}{4n}+1} = \frac{8}{27}n\left|_0^a\left(\frac{9x}{4n}+1\right)^{\frac{3}{2}} = \frac{8}{27}n\left[\left(\frac{9a}{4n}+1\right)^{\frac{3}{2}}-1\right].$$

For example, let the equation of the curve be $4y = x^{\frac{3}{2}}$, then $n = 16$, and the length of the arc between the limits of $x = 0$ and $x = a = 4$ is $122/27 = 4.5189$. Whether the story is true or not, the length of this curve can certainly be algebraically expressed and be computed by simple arithmetic.

## 118

Another curve whose length is expressed by simple algebra is the cycloid. This curve is generated by a point on the circumference of a wheel which rolls along the straight line $DE$. Thus the point $A$ in Fig. 51 reaches the horizontal line at $E$ when the circle has made half a revolution and in its progress the semi-cycloid $APE$ is described. Let $a$ be the radius $CA$ of the generating circle, and $P$ be any point on the cycloid whose coördinates are $x$ and $y$, the latter being measured downward. Then the length of the curve $AP$ is $\sqrt{8\,ay}$ and the length of $AE$ is $8\,a$, expressions of the greatest simplicity. The area between the cycloid $DAE$ and the straight line $DE$ is three times the area of the generating circle or $3\,\pi a^2$. The cycloid has also interesting properties which will be mentioned later under Mechanics.

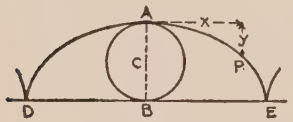


Fig. 51

## 119

The lengths of some curves of pursuit are also algebraically expressible. The simplest case (Fig. 52) is where the hare starts at $O$ and runs with uniform speed $v$ on the axis $OY$ while the dog starts at a point $A$ on the $X$-axis and runs always directly toward the hare with the speed $V$. When the dog is at $P$ the hare is at $Q$ and the tangent to the curve of pursuit is $PQ$. Let $a$ be the distance between the initial positions $O$ and $A$, and let $n$ be the ratio of the speeds

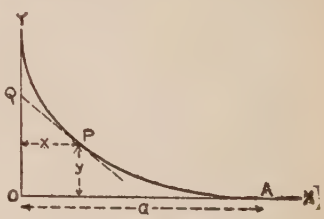


Fig. 52

$v/V$. Then the equation of the curve of pursuit, when $n$ is not equal to unity, is

$$y = \frac{a^n x^{1-n}}{2(n-1)} + \frac{x^{1+n}}{2a^n(n+1)} + \frac{an}{1-n^2}.$$

For example, let the dog run twice as fast as the hare, or $n = \frac{1}{2}$, then the equation of the curve is

$$y = \frac{x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} - a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{2}{3}a.$$

The length of an element of the curve being $dx\sqrt{1+p^2}$, where $p = dy/dx$, the length of the curve from $A$ to $P$ is found to be algebraically expressed, thus:

$$\int_x^a \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \frac{4}{3}a - a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3a^{\frac{1}{2}}}.$$

When $x = 0$ then $y = 2/3\,a$, and the length of the curve is $4/3\,a$. Here the dog has run double the distance that the hare has run, and it catches the hare at the point $x = 0$, $y = 2/3\,a$.

When $n$ is equal to or greater than unity, the dog can never catch the hare. When $n$ is less than unity the dog will catch the hare. The student in calculus may find it profitable to solve the following problem: Let the dog run 10 feet per second and the hare 8 feet per second, and let $a = 720$ feet; prove that the dog will catch the hare in 6 minutes and 40 seconds from the instant when the hare starts at $O$ and the dog starts at $A$.

## CHAPTER VII

## ASTRONOMY AND THE CALENDAR

### 120

STRONOMY is probably the most ancient of the physical sciences, the first facts being observed by shepherds who watched their flocks at night. The historian Josephus, in his Antiquities of the Jews, begins with the creation of the world and follows closely the biblical narrative. Speaking of Phaleg, fourth in descent from Noah and of his son Tera, who was the father of Abraham, he says: "God afforded them a longer life on account of their virtue and the good use they made of it in astronomical and geometrical discoveries." Speaking of the sojourn of Abraham among the Egyptians, he says, "He communicated to them arithmetic and delivered to them the science of astronomy; . . . they were unacquainted with those parts of learning, for that science came from the Chaldeans into Egypt and from thence to the Greeks."

### 121

The order of the twelve constellations of the zodiac may be remembered by the following ancient lines:

The Ram, the Bull, the Heavenly Twins,
Next the Crab, the Lion shines,
The Virgin, and the Scales,
The Scorpion, Archer, and the Goat,
The man who holds the watering Pot,
And Fish with glittering tails.

In memorizing this it is well to note, that the word shines should rhyme with Twins, and Pot with Goat.

The order here is from west toward east; when the Ram is setting in the west the Scales are rising in the east, when the Scales are setting in the west the Fish are rising in the east. This is a rough statement only, for at certain seasons of the year less than one-half of these constellations are above the horizon, while at other seasons more than one-half of them are visible at one time. Unfortunately the artist who put several of the constellations of the zodiac on the ceiling of the grand concourse in the Grand Central Station in New York, reversed this order, for there we see Aquarius in the east while the Crab is in the west. The copy from which he worked evidently had been incorrectly made; perhaps he took it from a celestial globe and then turned it around so as to interchange east and west.

## 122

The greatest of all optical instruments was the reflecting telescope of William Herschel, which was finished in 1789. The tube was forty feet in length, five feet in diameter, and weighed 60,000 pounds. With this telescope, magnifying 6450 times, he discovered two new moons circling around the planet Saturn, and recorded hundreds of new double stars and nebulæ. His sister, Caroline Herschel, was his constant companion in all his astronomical labors.

William Herschel died in 1822. In 1839 his celebrated son, John Herschel, took down the great telescope, which had then become a victim to the ravages of time and could no longer be used. The long tube was carefully laid upon three stone pillars where it could be preserved as a relic of the past. In the Christmas holidays of that year, John

Herschel, his wife, and their six children held a family feast in the great tube, and there they sang a song written by him in honor of the occasion:

In the old telescope's tube we sit,
And the shades of the past around us flit,
His requiem we sing with shout and din
As the old year goes out and the new year comes in.
Merrily, Merrily, let us all sing,
And make the old telescope rattle and ring.

Full fifty years did he laugh at the storm,
And the blast could not shake his majestic form.
Now prone he lies where he once stood high
And searched the heavens with his broad bright eye.
Merrily, Merrily, etc.

Here watched our father the wintry night
And his gaze was fed by pre-adamite light;
His labors were lighted by sisterly love,
And united they strained their vision above.
Merrily, Merrily, etc.

## 123

Galileo was the first man who looked at the heavenly bodies through a telescope. It was in 1610 that he saw four satellites moving around the planet Jupiter, and this demolished the theory that the earth was the center around which the planets revolved. These four satellites of Jupiter were the only ones known until 1892, but since then four smaller ones have been discovered. The earth has one moon, Mars has two, Jupiter has eight, and Saturn has nine or ten. The inner moon of Mars is near the planet and has such a high velocity that it rises in the west and sets in the east, while both new and full moon can be observed in a single night. All other known satellites, like our own moon, rise in the east and set in the west.

## 124

### BOTANY AND ASTRONOMY

If we examine the leafy stem of a plant we shall find the leaves upon it arranged in a symmetrical order and in a way uniform for each species. If a line be drawn around the stem from the base of one leaf stalk to that of the next, and so on, this line will wind around the stem as it rises, and on any particular plant there will be the same number of leaves for each turn around the stem. In the basswood, the Indian corn, and all the grasses, we have the two-ranked arrangement; the second leaf starting on exactly the opposite side of the stem from the first, the third opposite the second and hence directly over the first, so that all the leaves are in two vertical ranks, one on one side of the stem and one on the other. Next is the three-ranked arrangement such as is seen in sedges; here the second leaf is one-third of the way around the stem, the third one two-thirds, and the fourth one directly over the first. Then in the apple, cherry, and most of our common shrubs, the leaves are arranged in five vertical ranks, and the spiral winds twice around the stem before it reaches a leaf directly over the first one; here the distance between any two ranks is two-fifths of the circumference of the stem. Then in the common plantain there are eight ranks, and three turns around the stem, so that the distance between any two ranks is three-eighths of the circumference.

Now if we express these arrangements by figures, we have the fractions 1/2, 1/3, 2/5, 3/8, in which the denominator expresses the number of ranks and the numerator the number of turns of the spiral line around the stem before it reaches a leaf directly above the one from which it started.

Thus 1/2 stands for the two-ranked arrangement where there are two turns. But we notice that the numerator of any fraction is equal to the sum of the numerators in the two preceding fractions, and that the same is true for the denominators. Then the next fraction after 3/8 will be found by adding 2 and 3 for its numerator and 5 and 8 for its denominator, which gives 5/13. Thus we have the following series, 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34, etc., and just such arrangements of leaves are found, and no others. The fraction 5/13 gives the law for the common house leek, the others are found in the pine family and in many small plants.

The furthermost planet from the sun is Neptune, then follow Uranus, Saturn, Jupiter, the Asteroids, and Mars, then the Earth, Venus, and Mercury. Neptune makes its revolution around the sun in about 60,000 days, Uranus in 30,000 days or 1/2 the time of Neptune; in like manner Saturn's period is nearly 1/3 of that of Uranus, Jupiter's period is 2/5 that of Saturn, and so on until we come to the earth, following closely the same series as given above for the leaves on a stem. Thus the mathematical expression of the arrangement of the leaves of plants is approximately the same as that of the periods of the exterior planets. These arrangements of leaves ensure to plants a better distribution of the light and heat of the sun; the periods of the planets render them stable under the laws of gravitation. Perhaps the botanist, had he known that these figures apply both to leaves and planets, might have foretold the discovery of the asteroids or announced the existence of Neptune.

## 125

### THE MOON HOAX

In 1833 Sir John Herschel sailed from England for the Cape of Good Hope, carrying a large telescope with which to view the southern stars. This was before the times of steamboats and telegraphs so that more than two years passed away before any definite account of the discoveries of Sir John reached England or America. In 1835 the *New York Sun* published a series of articles, entitled "Great Astronomical Discoveries made by Sir John Herschel at the Cape of Good Hope." In the first article was given a circumstantial and highly plausible account as to how this early and exclusive information had been obtained by the paper. Then comes an interesting account of the inception and construction of the great telescope which he carried to the Cape. The idea of great magnifying power originated, it was said, in a conversation with Sir David Brewster regarding optics. "The conversation became directed to that all-invincible enemy, the paucity of light in powerful magnifiers. After a few minutes silent thought Sir John diffidently inquired whether it would not be possible to effect a transfusion of artificial light through the focal object of vision. Sir David, somewhat startled at the originality of the idea, paused awhile, and then hesitatingly referred to the refrangibility of rays and the angle of incidence. Sir John continued, 'why cannot the illuminated microscope, say the hydro-oxygen, be applied to render distinct, and if necessary even to magnify the focal object?' Sir David sprang from his chair in an ecstasy of conviction, and leaping halfway to the ceiling exclaimed, 'Thou art the man!'"

The interest of the reader being thus aroused by this

imaginary scientific conversation, the article goes on to describe the great telescope which was shipped to the Cape and there drawn by two relief teams of oxen to the place where it was erected. This place was "a perfect paradise in rich and magnificent mountain scenery, sheltered from all winds and where the constellations shone with astonishing brilliancy." Here Sir John observed stars and nebulæ, but above all he paid particular attention to the Moon. The magnifying power of his telescope was 42,000 times, so that objects on the Moon could be seen as if only six miles away and an object only 18 inches in diameter could be plainly recognized. Hence he clearly saw on the moon "basaltic rock, forests, and water, beaches of brilliant white sand girt with castellated marble rocks." He beheld herds of brown quadrupeds of the bison kind, each animal having a hairy veil over its eyes, and he conjectured "that this was a providential contrivance to protect the eyes from the great extremes of light and darkness to which all beings on the moon are periodically subjected." He also saw a species of beaver which was acquainted with the use of fire as was evident from the smoke that occasionally rose from their habitations.

Finally, of course, his search was rewarded by the sight of human beings with wings and who walked erect and dignified when they alighted on the plain. "They appeared in our eyes scarcely less lovely than the representations of angels by our more imaginative schools of painters; their works of art were numerous and displayed a proficiency of skill quite incredible to all except actual observers."

This hoax was immediately swallowed by the general public and caused much discussion. The *Sun* issued in pamphlet form an edition of 60,000 copies which were sold

in less than a month, and translations of it were made in Europe. In 1859 a second pamphlet edition was issued in New York with illustrations of the moon and with added notes.

The author of this most entertaining and successful hoax was Richard Adams Locke, then editor of the *Sun*. He was engaged in newspaper work for a large part of his life and died in 1871 at his home on Staten Island. An obituary notice describes him as "a warm-hearted man, well read, enthusiastic, and sometimes very eloquent on paper. His habits were rather convivial, but he was just and fearless, full of the best intentions, and overflowing with original inspirations."

## 126

The planet on which we live claims, of course, a large share of our attention. In 1878 Americus Symmes published his "Theory of Concentric Spheres," demonstrating that the earth is hollow, habitable within, and widely open about the poles. It contains arguments drawn from the statements of explorers in the polar regions, from the dip and variation of the magnetic needle, from the migrations of fish, from the spots on the sun and from the rings of Saturn. According to this remarkable theory, there are two openings at the poles into the hollow earth, the diameter of the northern one being about 2000 miles while the southern one is somewhat larger; the planes of these openings are parallel to each other but they make an angle of 12 degrees with the equator. Capt. Symmes imagined that the crust of the earth is about a thousand miles in thickness but he wisely refrained from giving any account of what is found within the hollow sphere.

## 127

Perhaps the earliest mention of a sun dial is that found in II Kings, xx, 9–11:

9. And Isaiah said, This sign shalt thou have of the LORD, that the LORD will do the thing that he hath spoken; shall the shadow go forward ten degrees or back ten degrees?

10. And Hezekiah answered, It is a light thing for the shadow to go down ten degrees: nay, but let the shadow return backward ten degrees.

11. And Isaiah the prophet cried unto the LORD: and he brought the shadow ten degrees backward, by which it had gone down in the dial of Ahaz.

On a properly constructed sun dial, such as is described below, the shadow cannot go backward. But a dial having a vertical style or gnomon when tilted from the horizontal possesses the property that the shadow will travel backward for a short time near sunrise and sunset. A dial with a vertical gnomon is, however, quite useless in telling the time of day.

## 128

### THE SUN DIAL

Four or five hundred years ago the only way to tell the hour of the day was by looking at a sun dial, for clocks and watches had not then come into use. Fig. 53 shows such a dial, which indicates 2 P. M. by the edge of the shadow cast upon the graduated surface by an inclined gnomon. Of course the sun dial is useless on a cloudy day, but when the sun does shine it gives apparent solar time with a probable error of about ten minutes, which is sufficiently close for the purposes of agriculture. A sun

Fig. 53

dial is usually placed in a horizontal plane, but in olden times they were often put on the walls of churches and public buildings, and many such can be seen in Europe even at this day.

The board on which the lines of the sun dial are drawn may be of any shape, but in Fig. 54 it is indicated as rectangular. This board is to be placed horizontally with its central line *NS* coinciding with the meridian of the place and is usually observed from its southern side. The shadow of the gnomon *AB* falls toward the western side in the morning and toward the eastern side in the afternoon. The lines which radiate from the center *A* being properly drawn the observer will see the shadow coinciding with the line 8 at eight o'clock in the morning, with the line 12 at noon, and with the line 3 at three o'clock in the afternoon. When the shadow is one-fourth of the distance from the line 3 to the line 4, the sun time is 3.15 P. M.

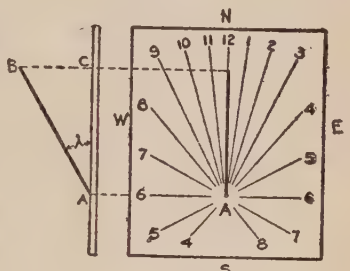


Fig. 54

The gnomon *AB* must be inclined to the plane of the board at an angle equal to the latitude of the place. Sometimes this is a thin sheet of metal *ABC* fastened onto the board, the edge *AB* being the true gnomon; sometimes it is a small metal rod *AB*, the end *B* being supported by another rod *BC*. It is essential that the inclination of *AB* to a horizontal dial plate must be equal to the latitude of the place, or for a dial in any position *AB* must point to the celestial pole.

How to make a horizontal dial: On a smooth board draw the lines *NS* and *EW*. The northern end of the line *NS* is to be numbered 12 for twelve o'clock noon, and the ends *EW* are to be numbered 6 for 6 A. M. and 6 P. M. The latitude of the place may be taken from a good map with sufficient precision for the construction of a sun dial, or if great precision is required it may be found by an astronomical observation. This latitude $\lambda$ is to be used for constructing the gnomon, and also for computing the angles which the radiating lines of the dial make with the central line *NS*.

To find the angle $\alpha$ which any radiating line makes with the central line *AN*, let $n$ be the number of hours before or after noon when the shadow should fall on that line; then

$$\tan \alpha = \sin \lambda \tan n\, 15^\circ.$$

Accordingly, the values of $\tan \alpha$ are as follows:

For 1 and 11 o'clock, $n = 1$ and $\tan \alpha = 0.268 \sin \lambda$.
For 2 and 10 o'clock, $n = 2$ and $\tan \alpha = 0.577 \sin \lambda$.
For 3 and 9 o'clock, $n = 3$ and $\tan \alpha = 1.000 \sin \lambda$.
For 4 and 8 o'clock, $n = 4$ and $\tan \alpha = 1.732 \sin \lambda$.
For 5 and 7 o'clock, $n = 5$ and $\tan \alpha = 3.732 \sin \lambda$.
For 6 and 6 o'clock, $n = 6$ and $\tan \alpha = \infty$.
For 7 and 5 o'clock, $n = 7$ and $\tan \alpha = -3.732 \sin \lambda$.

Now the values of $\alpha$ will be different for different latitudes. The following table gives values of $\alpha$ for three latitudes which have been computed from the above formulas with the help of a trigonometric table.

| | $\lambda = 30$ | $\lambda = 40$ | $\lambda = 50$ |
|---|---|---|---|
| For 1 and 11 o'clock | $\alpha =$ 7° 38′ | 9° 46′ | 11° 36′ |
| For 2 and 10 o'clock | $\alpha =$ 16 06 | 20 22 | 23 52 |
| For 3 and 9 o'clock | $\alpha =$ 26 33 | 32 44 | 37 27 |
| For 4 and 8 o'clock | $\alpha =$ 40 54 | 48 04 | 53 00 |
| For 5 and 7 o'clock | $\alpha =$ 61 49 | 67 23 | 70 43 |
| For 6 and 6 o'clock | $\alpha =$ 90 00 | 90 00 | 90 00 |
| For 7 and 5 o'clock | $\alpha =$ 118 11 | 112 37 | 109 17 |

When the radiating lines have been drawn, the gnomon put in place, and the board neatly painted, the sun dial is ready for erection. The board must be placed duly level with its *NS* line coinciding with the true meridian, and then, when the sun shines, delighted spectators may compare apparent solar time with their watches and wonder at the scientific skill of the youth who constructed the sun dial.

The largest sun dial ever built is at the royal observatory in Jaipur, India; it was erected about 1750 by the Mahárája Siwái Jai Singh II. Its gnomon is about 175 feet long and this can be ascended by stairs. The shadow of the gnomon falls on a large stone quadrant of 50 feet radius along which it moves at the rate of $2\frac{1}{2}$ inches per minute. Jaipur is in latitude 27 degrees north.

129

Southworth, in his "Four Thousand Miles of African Travel" (New York, 1875) gives a novel method of determining the true meridian: "The Arab when he prays, kneels toward Mecca. It is said that even the youngest never fails to bend, almost accurately, in that direction. Thus, in the form of living flesh, we had the Arab, by whom to find the variation of the compass; and, with the corrected bearing, we could find, when the sun bore due south or otherwise, the true meridian, and consequently noon."

130

The civil day begins at sunset among the Mahomedans and at midnight in Christian countries and is divided into twenty-four hours. The sun dial has been used from a remote antiquity to indicate apparent solar time. Clocks with wheels were devised about 1250 but they did not come

into general use until after 1600. It was found that these clocks at some times of the year were slower and at other times faster than apparent solar time. An accurate clock or watch keeps mean solar time, this being the time which would be indicated on a sun dial if the sun were perfectly uniform in his apparent motion throughout the year. The difference between apparent and mean solar time is called the Equation of Time and its values are given in some almanacs under the headings "clock slow" or "clock fast." The following table shows such values to the nearest minute which are to be added to apparent time (or subtracted when marked −) in order to give mean or clock time.

| | | | | | |
|---|---|---|---|---|---|
| 1 Jan. | 3 min. | 1 May | −3 min. | 1 Sept. | 0 min. |
| 10 Jan. | 8 min. | 10 May | −4 min. | 10 Sept. | 3 min. |
| 20 Jan. | 11 min. | 20 May | −3 min. | 20 Sept. | − 6 min. |
| 1 Feb. | 14 min. | 1 June | −2 min. | 1 Oct. | −10 min. |
| 10 Feb. | 14 min. | 10 June | −1 min. | 10 Oct. | −13 min. |
| 20 Feb. | 14 min. | 20 June | 1 min. | 20 Oct. | −15 min. |
| 1 Mar. | 12 min. | 1 July | 4 min. | 1 Nov. | −16 min. |
| 10 Mar. | 11 min. | 10 July | 5 min. | 10 Nov. | −16 min. |
| 20 Mar. | 8 min. | 20 July | 6 min. | 20 Nov. | −14 min. |
| 1 Apr. | 4 min. | 1 Aug. | 6 min. | 1 Dec. | −11 min. |
| 10 Apr. | 1 min. | 10 Aug. | 5 min. | 10 Dec. | − 7 min. |
| 20 Apr. | −1 min. | 20 Aug. | 3 min. | 20 Dec. | − 2 min. |

This table will be useful when one compares his watch with a sun dial. As all the affairs of life are now regulated by clock time, it also explains why the time of sunset appears to rapidly become earlier in October and to rapidly become later in January.

The apparent and mean solar time above described is different for places having different longitudes, and in general may be designated as local time. In recent years, owing to the requirements of railroad operation, most

clocks and watches keep standard time or the local time on a certain meridian. In the United States there are four standard meridians, those of longitude 75°, 90°, 105°, and 120° west of Greenwich. Eastern standard time is mean solar time of the 75° meridian, central standard time is mean solar time of the 90° meridian, mountain standard time is mean solar time of the 105° meridian, and Pacific standard time is mean solar time of the 120° meridian. In going from one of these meridians to the next one, our watch must be set one hour backward or forward according as we go west or east.

When a watch keeping standard time is read at a place which is one degree of longitude west of the standard meridian it is four minutes faster than mean local time of that place; when the place is two degrees to the westward the watch is eight minutes faster, for three degrees westward twelve minutes faster and so on. When read at places to the eastward it is four minutes slower for each degree of longitude. Hence, this must be taken into account also when comparing a watch with a sun dial.

## 131

### DAYS, MONTHS, AND YEARS

Julius Cæsar, with the help of the astronomer, Sioseneges, introduced the method of reckoning known as the Julian calendar. The year being 365.2422 solar days, he took 365 such days for a common year and 366 days for a leap year, so that the average length of a year was 365.25 days. This Julian calendar is still in use in Russia and Greece, but it was supplanted in most of Europe in 1582 by the Gregorian calendar. In the Julian calendar all years divisible by 4 were leap years; in the Gregorian calendar years

divisible by 4 are leap years unless they are divisible by 100 and not by 400. Thus, in the Gregorian calendar the years 1600 and 2000 are leap years, but the years 1700, 1800, 1900 are common years. In 1582 the Julian calendar was ten days slower than the Gregorian, after 1700 it became eleven days slower, and since 1900 it has been thirteen days slower. Hence, Jan. 1, 1917 of the Gregorian calendar corresponds to Dec. 19, 1916 of the Julian.

**1752** **September hath XIX Days this Year.**

First Quarter, the 15th day at 2 afternoon.
Full Moon, the 23rd day at 1 afternoon.
Last Quarter, the 30th day at 2 afternoon.

| M D | W D | Saints' Days Terms, &c. | Moon South | Moon Sets | Full Sea at Lond. | Aspects and Weather |
|---|---|---|---|---|---|---|
| 1 | f | Day br. 3.35 | 3 A 27 | 8 A 29 | 5 A 1 | ☊ ☋ ☿ |
| 2 | g | London burn. | 4 26 | 9 11 | 5 38 | Lofty winds |

According to an act of Parliament passed in the 24th year of his Majesty's reign and in the year of our Lord 1751, the Old Style ceases here and the New takes its place; and consequently the next Day, which in the old account would have been the 3d is now to be called the 14th; so that all the intermediate nominal days from the 2d to the 14th are omitted or rather annihilated this Year; and the Month contains no more than 19 days, as the title at the head expresses.

| | | | | | | |
|---|---|---|---|---|---|---|
| 14 | e | Clock slo. 5 m. | 5 15 | 9 47 | 6 27 | Holy Rood D. |
| 15 | f | Day 12 h. 30 m. | 6 3 | 10 31 | 7 18 | and hasty |
| 16 | g | | 6 57 | 11 23 | 8 16 | showers |
| 17 | A | 15 S. Aft. Trin. | 7 37 | 12 19 | 9 7 | |
| 18 | b | | 8 26 | Morn. | 10 22 | More warm |
| 19 | c | Nat. V. Mary | 9 12 | 1 22 | 11 21 | and dry |
| 20 | d | Ember Week | 9 59 | 2 24 | Morn. | weather |
| 21 | e | St. Matthew | 10 43 | 3 37 | 0 17 | ♂ ♀ ☿ |
| 22 | f | Burchan | 11 28 | ☽ rise | 1 6 | ☊ ☋ ♀ |
| 23 | g | Equal D. & N. | Morn. | 6 A 13 | 1 52 | ♂ ☉ ♂ |
| 24 | A | 16 S. Aft. Trin. | 0 16 | 6 37 | 2 39 | ♂ ☉ ☿ |
| 25 | b | | 1 5 | 7 39 | 3 14 | |
| 26 | c | Day 11 h. 52 m. | 1 57 | 8 39 | 3 48 | Rain or hail |
| 27 | d | Ember Week | 2 56 | 8 18 | 4 23 | ♂ ♂ ☿ |
| 28 | e | Lambert bp. | 3 47 | 9 3 | 5 6 | now abouts |
| 29 | f | St. Michael | 4 44 | 9 59 | 5 55 | ✶ ♄ ♀ |
| 30 | g | | 5 43 | 11 2 | 6 58 | |

In Great Britain and its colonies the change of the Julian to the Gregorian calendar was not made until 1752. In September of that year eleven days were omitted from the almanacs. The above is a copy of the calendar for September, 1752, taken from the Almanac of Richard Saunders, Gent., published in London. All English and American almanacs gave similar statements for that month. The Ladies' Diary or Woman's Almanac indulged in poetry, appropriate to the occasion:

> The third of September the fourteenth is nam'd,
> For which British annals will ever be fam'd.
> For by Wisdom and Art to the House made appear
> The Sun was reduc'd to attend on the Year;
> His Julian vagaries long time has he known,
> But has now got a new bridal Year of his own

132

In both Julian and Gregorian calendars the months are those established by Julius Cæsar, namely:

> Thirty days hath September,
> April, June, and November,
> All the rest have thirty-one,
> Save in February which, in fine,
> In common years, hath twenty-eight,
> And in leap years twenty-nine.

The time when the year began has been different in different countries. In Cæsar's reign it appears that March was the first month; thus September was the seventh and December the tenth, as the names imply. The early English almanacs, however, begin the year with January as at present, but the legal year of the British government began on March 25, although March was called the first

month. In legal and church records prior to 1752, it is common to find dates like Feb. 20, 1695, or Feb. 20, 1695/6, these being intended for the historical or almanac year 1696.

## 133

A very convenient rule for determining the day of the week corresponding to the day of the month in any year was given by Prof. Comstock in *Science*, Nov. 18, 1898. Let $Y$ be any year of the Gregorian calendar and $D$ the day of the year. Divide $Y - 1$ by 4, by 100, and by 400, neglecting the remainder in each case. Then find $S$ from

$$S = Y + D + \frac{Y-1}{4} - \frac{Y-1}{100} + \frac{Y-1}{400}$$

and divide $S$ by 7; the remainder gives the day of the week, 0 indicating Saturday, 1 Sunday, 2 Monday, and so on. For example, take July 4, 1916; here $Y = 1916$, $D = 186$, $(Y-1)/4 = 478$, $(Y-1)/100 = 19$, $(Y-1)/400 = 4$. Then $S = 2565$, and this divided by 7 gives a remainder 3; hence, July 4, 1916 comes on Tuesday. The reason for this rule is clear, if it be remembered that all years exactly divisible by 4 are leap years except when they are even century years, as 1800, 1900, 2000, etc., when they must be divisible by 400; thus the subtractive term $(Y-1)/100$ prevents the addition of an extra day during such years as 1800, 1900, and 2100, while it also makes only one extra day to be added during the year 2000. Of all the rules for finding the day of the week from a given day of the month and year, this is by far the simplest.

For the Julian calendar the following rule may be used to find the day of the week corresponding to a given date.

Let $Y$ be the year and $D$ the day of the year. Neglecting the remainder in the fractional term, compute $S$ from

$$S = Y + D + \frac{Y - 1}{4} - 2$$

and divide $S$ by 7; then the remainder gives the day of the week, 0 indicating Saturday, and so on. For example, Columbus discovered America on Oct. 12, 1492; here $Y = 1492$, $D = 286$, $(Y - 1)/4 = 372$, then $S = 2148$ which divided by 7 gives 6 for a remainder; hence America was discovered on a Friday. Again George Washington was born on Feb. 11, 1732; here $Y = 1732$, $D = 42$, $(Y - 1)/4 = 432$, and then $S = 2204$ which divided by 7 gives 6 for a remainder; hence, Washington was born on a Friday.

The common opinion that Washington was born on Feb. 22 is erroneous. This originated in the idea of irresponsible persons that Gregorian time ought to be extended backward into Julian time. This reprehensible idea is founded on no sound principle, and in celebrating the birthday of Washington on Feb. 22, we all commit grievous error.

## 134

J. W. Nystrom of Philadelphia devised about fifty years ago the "tonal system" of numeration in which 16 is the base instead of 10 as in the decimal system. The numerals 1, 2, 3, 4, etc., were called An, De, Ti, Go, etc., and new characters were devised for 10, 11, 12, 13, 14, 15. This system embraced also a new division of the year into 16 months, these having the names Anuary, Debrian, Timander, Gostus, Suvenary, Bylian, Ratamber, Mesidius, Nictorary, Kolumbian, Husander, Victorious, Lamboary, Folian, Fylander, Tonborious, the first two letters of each

month being the names of the sixteen numerals. Nystrom certainly did his work well.

### 135

Josh Billings in his almanac said that the name February was derived from a Chinese word which meant kondem cold. Josh was right regarding the temperature. At the head of the calendar for July he gives this verse:

Young man, let hornets be
And don't go nigh the pizen snake too much,
For in the month of July
They a'in't healthy to the touch.

For another month he gives this excellent advice:

He who by farming would get rich,
Must plow, and hoe, and dig, and sich,
Work hard all day, and sleep hard all nite,
Save every cent and not get tite.

For the first of April he has the following:

April Phool was born this day,
A simpleton, but clever,
And though 3000 years of age,
He's just as big a phool as ever.

### 136

Comets in ancient times brought great mental distress upon people, for they were supposed to presage war, famine, or pestilence. Even to astronomers the phenomena of the tail being repelled by the sun backward from the nucleus of the comet has been a great mystery, for it seemed to contradict the law of universal attraction. Now, however, we understand that the small particles of the tail are driven away from the head by the pressure exerted by the light of the sun, so that the mystery appears to have been solved.

Yet even at this day the appearance of a comet incites a feeling of awe, and the words of the poet Holmes arise in the memory:

The Comet! He is on his way,
And singing as he flies;
The whizzing planets shrink before
The spectre of the skies.
Ah! well may regal orbs burn blue,
And satellites turn pale,
Ten million cubic miles of head,
Ten billion leagues of tail!

## CHAPTER VIII

## MECHANICS AND PHYSICS

### 137

IT IS a misfortune that physicists and engineers teach to students two different systems of units. A boy comes to a technical school, understanding perfectly, from his experience, what is meant by force and what is meant by a force of ten pounds or ten kilograms. The teacher of physics tells him that forces must be measured in poundals or dynes, notwithstanding that no apparatus for measuring forces in such units has ever been made or used. The result is great mental confusion to the boy, from which he does not recover until he joins the class in engineering where he finds that forces are measured in those units to which he had always been accustomed before he entered upon the instruction of the physicist. All this might be avoided if mechanics were omitted entirely from courses in physics. Surely the subjects of heat, light, sound, and electricity furnish a sufficient field for the physicist, without encroaching on the topic of mechanics, which properly belongs to the engineer.

### 138

The unfortunate equation $F = mf$ comes early in a course in mechanics as taught by a physicist. Here the mass $m$ is measured in units of a standard lump of metal furnished by

the government; acting for one second on this lump is a force $F$, which produces the velocity $f$ at the end of that second. More generally $f$ is called the acceleration, or change in velocity in one second, and its unit is one unit of length per second. Let $L$ represent length in general and $T$ time, while $M$ represents mass, then we have $F = ML/T^2$, or force dimensionally equals mass multiplied by length divided by the square of time. The student tries hard to comprehend this, but finds it impossible, for he *knows* that force is not $ML/T^2$ and he *knows* that there is no way to measure a force except by the number of units of force which it contains.

The truth of the matter is that the equation $F = mf$ is not true. Experiments and experience teach that $mf$ is proportional to $cF$ where $c$ is a constant, not that $mf$ equals $F$. When there are two different forces $F$ and $G$ which act at different times on the same body they produce accelerations $f$ and $g$. Experience and experiments show that these forces are proportional to the accelerations which they produce, whence

$$F/G = f/g.$$

This is a fundamental equation which is entirely correct. If the teacher starts with this, his students will have no confusion of mind.

## 139

Into an apple cut two holes inclined like $ab$ and $cb$ in Fig. 55. Into each hole put a small quill so that when the string $AC$ is inserted the friction may be small. Then pull horizontally upon the string by its ends $A$ and $C$. As the pull increases, the apple will

Fig 55

be seen to rise vertically by the upward pressure of the string at *b*; as the pull slightly decreases, the apple will fall. The spectators, who think that the string passes straight through the apple, are filled with wonder at the strange motion of the apple bobbing up and down.

## 140

### CENTER OF GRAVITY

Many amusing mechanical tricks depend upon the principle that the center of gravity of a system of bodies always takes the lowest possible position; thus, a system will be stable if its center of gravity, when slightly disturbed, tends to fall to its original position.

To balance a cork upon the small end of a cane held vertically with that end upward. Put the prongs of two forks into opposite sides of the cork, letting the forks incline downward at angles of about 30 degrees with the vertical. Then the center of gravity of the cork and forks will be below the bottom of the cork, and thus there will be no danger of its falling off the end of the cane. The cane can be carried around held in a vertical position with the cork thus balanced on it.

A cork with two forks thus attached may be made to walk along a horizontal bar. Put two pegs of equal lengths into the bottom of the cork to act as legs, one being slightly in advance of the other. Then place the cork with its forks upon the horizontal bar, and set the forks into oscillation like a pendulum, the oscillations being parallel to the plane of the bar. The cork will then be alternately supported upon one of the two legs, and hence will advance or walk along the bar as long as the oscillations continue.

## 141

When the vertical line through the center of gravity lies without the base of support, the body will fall over, but when it lies within the base it will not fall. A toy horse standing with only his hind feet on the edge of a table will not fall if a curved wire attached to his breast runs backward and has a ball of sufficient weight at the free end. The horse may be made to rock to and fro without danger of falling, if the center of gravity of the horse and ball always rises when disturbed and if the vertical line through that center does not fall beyond the edge of the table.

## 142

### INERTIA

Take several of the round wooden pieces which are used in playing checkers and put them in a vertical pile on a table. Then with a heavy knife blade strike the lowest block very quickly in a direction exactly parallel to the surface of the table. The lowest block will then move out under the impact of the blow but those above it will not be disturbed except that the whole pile will fall vertically to the table. This is an illustration of the doctrine of inertia, for there is no reason why the pile should move laterally unless it receives some impact from the blow; but this does not occur owing to the slight friction between the wooden pieces and to the suddenness with which the force is applied.

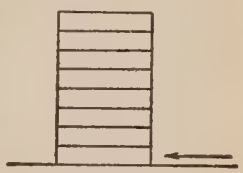


Fig. 56

The principle of inertia is utilized by the Japanese in a simple device (Fig. 57) for preventing the overthrow of

their pagodas by earthquakes. From the roof $A$ of the pagoda there is suspended a heavy ball $B$ by a wooden pendulum rod. When the earthquake comes the foundation of the pagoda is moved laterally to and fro and with it the lower part of the walls. The ball $B$, however, does not move until the motion can be communicated to it from the roof through the suspending rod. As this is a slow process the top $A$ of the pagoda suffers only a slight lateral motion, and hence the structure is prevented from being overturned by the earthquake.

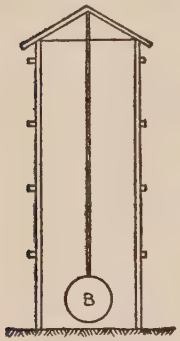


Fig. 57

The seismograph used for recording vibrations due to earthquakes depends upon a similar principle. A heavy ball is so arranged, usually at the end of a horizontal pendulum, that it remains practically at rest while the ground moves laterally from the earthquake shock. Attached to the ball is a pointer touching lightly a sheet of paper on the recording apparatus which rests on the ground or floor. As this paper moves to and fro, the stationary pencil traces a curve which shows the intensity and duration of the earthquake shocks.

## 143

The cause of inertia may be imagined to be a change in size or shape of the atoms of the body due to action of the ether. Thus when a force puts a body in motion the atoms assume new shapes or sizes and thus store up energy. When the moving body meets resistance this energy is expended in overcoming that resistance, and the velocity of the body decreases. When a body comes to rest it

cannot move again under the action of a force until the atoms have assumed new forms and thus stored up the energy imparted by the force.

## GRAVITATION

### 144

Gravitation is the great unsolved puzzle in the mechanics of the universe. The law of gravitation, namely, that any two atoms of matter attract each other with a force proportional to the product of their masses and inversely as the square of the distance between them, states merely observed facts and gives no clue as to the cause. The word attraction is perhaps an unfortunate one, for it implies that each body pulls upon the other. This might be true if each atom were joined to all other atoms by stretched elastic threads for the transmission of the force, but otherwise it is difficult to account for the force of pull. In fact, instances of pull are rare in mechanics; we say that the horse pulls the wagon, but in reality the horse pushes by his shoulders against the harness. The more rational explanation of gravity is that two bodies are pushed together by pressure exerted upon them from the space beyond their line of junction. To account for this push, LeSage supposed that multitudes of fine particles are moving in every direction through space. If there was only one body in the universe, these particles would impinge upon it from every direction and hence no motion of the body could occur. But for two bodies, it is plain that each will intercept particles that cannot fall upon the other, so that the bodies will be pushed together. While this accounts for the law of gravitation, it is of course no proof at all of the correctness of the theory, and there is no evidence at all of the fine moving particles.

Under the hypothesis of an ether which fills all space, the facts of gravitation require that bodies must be pushed together by the pressure of this ether. When two bodies are separated to a distance by applied forces, energy becomes stored in the ether; when the forces are removed this energy exerts pressures on each body which causes them to move toward each other. This general statement is about as far as we can go in explaining the cause of gravitation, but this rests upon the hypothesis of a universal ether, the existence of which has not been proved by any experimental facts.

**145**

Many absurd speculations regarding the cause of gravity have been made, and the following, from a pamphlet of 1893 called "Invisible and Visible," is one of the worst. "Gravitation is caused by the earth moving so fast that it draws everything to it, like a train of cars (when you stand close to the track) as it is passing."

Magnetic or electric action can be prevented from being propagated to a distance by screens of suitable material, but nothing has ever been discovered by which the action of gravitation can be screened off. The attraction of the earth acts with the same power upon a body, whether or not other bodies be interposed between it and the earth. Years ago it was recognized that the problem of flying would be solved if by any means a flying machine could be wholly or partially relieved from the attraction of the earth.

In 1847 Orrin Lindsay published at New Orleans a pamphlet entitled "Plan of Aerial Navigation, with a Narrative of his Explorations in the higher Regions of the Atmosphere and his wonderful Voyage around the Moon."

His "plan" consisted in annulling the force of gravity. Well-prepared steel, after being superficially coated, amalgamated with quicksilver, and then strongly magnetized, proved to be an impervious screen to gravitation. A hollow box made of these metal plates, rose from the earth; to cause it to descend, a hole was opened in the bottom; to cause it to move laterally, a hole was opened in the side. It is unnecessary to explain here his voyage to the moon.

About 1900 there was published a novel by Simon Newcomb called "His Wisdom the Defender," in which flights by a huge machine were made by its property of annulling the force of gravity. The inventor and owner made aerial voyages over the earth, and compelled the nations to disband their armies under the threat of dropping bombs which would blow their cities into nothingness. Thus this inventor, who was called "His Wisdom," inaugurated upon the earth a reign of universal peace.

## 146

One of the most interesting papers on the ether of space is that of DeVolson Wood in the London Philosophical Magazine of November, 1885. It is based on the known facts: (1) that the ether transmits light at a velocity of 186 300 miles per second; (2) that the ether transmits 133 foot-pounds of energy per second from the sun to each square foot of the earth's surface. His discussion leads to the conclusions (1) that the mass of a cubic foot of the ether at the earth's surface is $2 \times 10^{-24}$ pounds, (2) that the ether has an elasticity such that it exerts a pressure of $4 \times 10^{-8}$ pounds on each square foot of the earth's surface, (3) that the ether has the enormous specific heat of 4 600 000 000 000, so that to raise one pound of it 1° F. would require as much

heat as it would to raise 2 300 000 000 tons of water the same amount. This medium, says Wood, will be everywhere practically non-resisting and sensibly uniform in temperature, density, and elasticity. In one pound of it there is $10^{10}$ times the kinetic energy of a pound of gas.

## 147

## THE DIAPHOTE HOAX

From a Pennsylvania daily newspaper of Feb. 10, 1880.

A special meeting of the Monacacy Scientific Club was held on Saturday evening to listen to a paper by Dr. H. E. Licks on the diaphote, an instrument invented by him after nearly three years of study, and now so nearly perfected that he feels warranted in bringing some few of the results thus far attained to the notice of the public. There were present, besides many scientists of Eastern Pennsylvania, Prof. M. E. Kannick of the polytechnic school at Pittsburg, and Col. A. D. A. Biatic of the Brazilian corps of engineers, who is now in this country making extensive purchases of iron and steel. The meeting was called to order by the president, Prof. L. M. Niscate, who in introducing Dr. Licks made a few remarks, saying that he had had an opportunity to witness a few experiments with the diaphote, and he felt convinced that it would ultimately rank with the telephone, the phonograph, and the electric light as one of the most remarkable triumphs of science in the nineteenth century.

Dr. Licks prefaced his paper by saying that the idea of the invention was first suggested to his mind about three years before by reading accounts of some of the early experiments of Bell's telephone, and that a little later when Edison brought out the carbon instrument, his studies had become so far advanced as to assure him of its theoretic

possibility. By the telephone the sound of the human voice may be heard hundreds of miles away. Why, then, cannot light be transmitted in a similar manner, so that by the use of a connecting wire one may distinctly see the image of the object far removed? This, said Dr. Licks, was the form in which the inquiry first suggested itself to him nearly three years ago, and he felt gratified to be able to exhibit to the club this evening an instrument called the diaphote in which the practical realization of the idea had been in a great measure satisfactorily obtained. The word diaphote, from the Greek *dia* signifying through, and *photos*, signifying light, had been selected as its name, implying that the light travelled through or in the wire. Although popularly this might be imagined to be the case, it was really no more so than with sound in the telephone. There the sound waves strike a diaphragm that is set into vibration, and generates induced electricity in the wire, this causing corresponding vibrations in another distant diaphragm which reproduces similar sounds. In the diaphote, likewise, the waves of light from an object strike a peculiarly constructed mirror or speculum which is joined by a wire with another similar speculum; the image of an object in the first modifies the electric current in the wire and passing quickly onward to the receiving instrument produces there a secondary image. The intermediate wire, as in the telephone, may be hundreds of miles in length, yet such is the delicacy of the diaphotic plates that the transmitted image of a simple object is almost as distinct as the original, and Dr. Licks feels confident that after the removal of a few obstacles, of a mechanical nature only, the most complex forms will be reproduced with the strictest fidelity as to outline and color.

The diaphote consists of four essential parts, the receiving mirror, the transmitting wires, a common galvanic battery, and the reproducing speculum. Dr. Licks gave a detailed account of the experiments to determine the composition of the mirror and speculum. For the former he had finally selected an amalgam of selenium and iodide of silver, and for the latter an amalgam of selenium and chromium. The peculiar sensitiveness of iodide of silver and chromium to light has long been known and their practical use in photography suggested their application in the diaphote. It was found, however, after many experiments, that their action must be so modified that each ray of light should influence the electric current proportionally to its position in the solar spectrum, and the element selenium was selected as best adapted to this purpose. At first a small mirror was employed with only a single wire, but the images in the speculum were confused and indistinct so that it became necessary to make the mirror of pieces each about one-third of a square inch in area and each having a wire attached. In the diaphote exhibited by Dr. Licks to the club, the mirror was six by four inches in size, and there were 72 wires which were gathered together into one about a foot back of the frame, the whole being wrapped with insulating covering; and in reaching the receiving speculum each little wire was connected to a division similarly placed as in the mirror. From a galvanic battery wires ran to each diaphotic plate and thus a circuit was formed which could be opened or closed at pleasure. Dr. Licks explained how the light caused momentary chemical changes in the mirror which modify the electric current and cause similar changes in the remote speculum, this causing a similar image which may be readily seen or be thrown upon a screen by a second

camera. He explained how the proportions of selenium should be scientifically adjusted to the resistance of the electric current so as to avoid any blending of the reproduced images. This, he said, had been the problem which had caused him the most trouble and which at one time had seemed almost insurmountable.

At the close of the paper an illustration of the powers of the instrument was given. The mirror of the diaphote, in charge of a committee of three, was taken to a room in the lower part of the building, and the connecting wires were laid through the halls and stairways to the speculum on the lecturer's platform. Before the mirror, the committee held in succession various objects, illuminating each by the light of a burning magnesium tape, since the rays from gas are deficient in actinic power; simultaneously on the speculum appeared the reproduced images, which for exhibition to the audience were thrown on a screen considerably magnified. An apple, a penknife, and a trade dollar were the first objects shown; in the latter the outlines of the goddess of liberty were recognized and the date 1878 was plainly legible. A watch was held for five minutes before the mirror and the audience could plainly perceive the motion of the minute hand, but the motion of the second hand was not satisfactorily seen, although Prof. Kannick by looking into the speculum said it was there quite perceptible. An ink bottle, a flower, and a part of a theater handbill were also shown, and when the head of a little kitten appeared on the screen the club expressed its satisfaction by hearty applause.

After the close of the experiments the scientists congratulated Dr. Licks on his invention, and the president made a few remarks on the probable scientific and industrial

applications of the diaphote in the future. With telephone and diaphote it may yet be possible for friends far apart to hear and see each other at the same time, to talk, as it were, face to face. In connection with the interlocking switch system it may be used to enable the central office to see many miles of track at one time, thus lessening the liability to accident. In connection with photolithography it could be so employed that the great English papers could be printed in New York a few hours after their appearance in London. Our reporter also learned that Dr. Licks will lecture on the diaphote next week before the American Society of Arts, and that he will make definite arrangements for the manufacture of the instrument as soon as the seven patents for which he has applied are formally issued.

Within a week after the publication of the above article, it was copied in whole or in part by numerous papers throughout the United States, many commenting editorially on the great possibilities of the marvellous diaphote. Some papers said that sunlight would be transmitted by it from the sunny side of the earth to light the side which was in darkness. The *New York Times* said "the imagination almost fails before the possibilities of what the diaphote may yet accomplish." The only paper which recognized the article as a fake seems to have been the *New York World*, which said, "the hoax is a clever one and is interesting also as depending for its success upon the opposite of the mistake which was at the bottom of Locke's famous 'Moon Hoax'; it is the misuse of the word mirror in connection with the new 'invention' which has made the miracle of it so acceptable to the public." Within a month after the publication of the diaphote hoax, items appeared in the papers announcing the invention in Pittsburg of an instrument called the

"telephole" by which two persons at a distance could see each other as they talked over the telephone, and by which any written or printed document could be transmitted instantaneously to any distance. The inventors of this instrument, it was stated, had labored many years in making experiments and now success had been attained. While Dr. Licks used 72 wires, the Pittsburg inventors used but one, and their applications for patents were soon to be granted.

News of the diaphote soon spread to Europe, and in due time there came back to us stories of wonderful inventions there made. For instance in 1889, the news came that a young German, named Korzel, exhibited an instrument by which a person in one city could read a newspaper held before a receiving plate in another distant city. The secret of this marvellous instrument, it was said, lay in the sensitiveness of selenium to the effects of light, its electric conductivity changing with the color and intensity of the light which impinged upon the plate. Very curiously all the inventors of such instruments have used selenium since its properties were first utilized in the diaphote by Dr. Licks. Almost every year similar stories have appeared, the most recent being one which was published in the *New York Times* of May 29, 1914, in the form of a cable dispatch from London. This article states that on the previous day, Dr. A. M. Low, a well-known scientific investigator, lectured before the Institute of Automobile Engineers on "Seeing by Wire." For five years his experiments had been carried on and now he had attained such success that pictures were reproduced at a distance of four miles. His instrument "has a receiving screen consisting of a large number of cells of selenium, over which a ruler is moved rapidly by a small

motor worked with a current of high frequency and about 50 000 volts pressure. The receiver at the other end is made up of a series of telephone slabs of steel, through which the light passes." Perhaps Dr. Low is on the right track, and if his apparatus becomes a verity, then he should give proper credit to Dr. H. E. Licks by calling it the diaphote.

## 148

### THE ONE-HOSS SHAY

The secret of successful engineering construction is to make each part of a structure just as strong as the other parts, so that there can be no weak spot where failure may occur. Oliver Wendell Holmes wrote many years ago a delightful poem on this principle. It begins:

> Have you heard of the wonderful one-hoss shay
> That was built in such a logical way
> It ran a hundred years to a day?

The "shay" was supposed to have been built by a Deacon in Massachusetts who was resolved that it should be properly constructed.

> But the Deacon swore (as deacons do)
> It should be so built that it couldn't break down,
> "Fur," said the Deacon, " 'tis mighty plain
> That the weakest spot must stan' the strain,
> And the way to fix it, as I maintain, is only jest
> To make that place as strong as the rest."
>
> The wheels were just as strong as the thills,
> And the floor was just as strong as the sills,
> And the panels just as strong as the floor,
> And the whipple-tree neither less or more.
> And the back cross bar as strong as the fore
> And spring and axle and hub encore.

The chaise was designed to run exactly a hundred years, and so it did. When that time arrived a parson was riding in it and the catastrophe came.

All at once the horse stood still,
Close by the meetin'-house on the hill,
First a shiver, and then a thrill,
Then something decidedly like a spill,
And the parson was sitting on a rock,
At half-past nine by the meetin'-house clock.

You see of course, if you're not a dunce
How it went to pieces all at once,
All at once and nothing first,
Just like bubbles when they burst.

End of the wonderful one-hoss shay,
Logic is logic! That's all I say.

## CHAPTER IX

## APPENDIX

### 149

ONCE upon a time a man, after much labor, raised a number of two digits to the 31st power this containing 35 digits. Stating this fact to a lightning calculator, he was about to give the long number, when the calculator said that this was unnecessary and that the root was 13. How did he know this? Simply from having committed to memory a table of two-place logarithms and by making a rapid computation from them. Since the given power has 35 digits its logarithm lies between 34.00 and 35.00. Dividing these by 31 gives 1.09 and 1.13 as the logarithms of numbers between which the root must lie. Then, remembering that the logarithms of 12, 13, and 14 are 1.08, 1.11, and 1.15 the computer instantly saw that the required number must be 13. Hence, the man who computed that 34 059 943 367 449 284-484 947 168 626 829 637 was the 31st power of 13 had his labor for his pains, for there was no opportunity to give a single figure of it to the lightning calculator. In fact 13 is the only number of two digits whose 31st power has 35 digits.

### 150

#### MERSENNE'S NUMBERS

In 1644 Pere Mersenne made certain statements regarding numbers of the form $2^p - 1$ where $p$ is a prime. These statements seem to be that the only values of $p$, not greater

than 257, which make $2^p - 1$ a prime, are 1, 2, 3, 5, 7, 13, 17, 31, 61, 127, 257 and that it is composite for all other values of $p$. Thus, $2^{11} - 1 = 2047 = 23 \times 89$, and $2^{23} - 1 = 2\,388\,607 = 47 \times 178\,481$. How he arrived at these conclusions is a mystery, but it is supposed to have been through correspondence with the great mathematician Fermat.

There are 56 primes not greater than 257. Mersenne's statement has been verified for 38 of these, namely, for 10 of the twelve values of $2^p - 1$ which he stated to be prime, and for 28 of the 44 values which he stated to be composite. Although much acute thought has been spent upon them by great mathematicians like Euler and Gauss, yet 18 values of $2^p - 1$ are yet unverified, namely, for $p = 89, 101, 103, 107, 109, 127, 137, 139, 149, 157, 167, 173, 193, 199, 227, 229, 241, 257$.

Fermat, in 1679, gave a rule for determining factors of the number $2^p - 1$. He said, in effect, that if 2 or 8 or 32 be subtracted from a perfect square, the remainder $n$ will generally divide $2^p - 1$ when $n$ is a prime and $n - 1$ is a multiple of $p$. Thus, from 25 take 2, the remainder 23 divides $2^{11} - 1$ since 23 is prime and $23 - 1$ is a multiple of 11. From 49 take 2, the remainder 47 divides $2^{23} - 1$ since 47 is prime and $47 - 1$ is a multiple of 23. From 225 take 2, the remainder 223 divides $2^{37} - 1$ since 223 is prime and $223 - 1$ is a multiple of 37. These three illustrations of the process are given by Fermat.

The reason for this rule is unknown to me, but following the same line of procedure I take 2 from 169 and 167 is known to be a factor of $2^{83} - 1$; also taking 2 from 361 the remainder 359 is a factor of $2^{179} - 1$; also taking 2 from 441 the remainder 439 is a factor of $2^{73} - 1$. Further, taking 32 from 121 the remainder 89 is known to be a factor of

$2^{11} - 1$. But this method breaks down when 32 is taken from 841; here the remainder is 809 and is prime and 808 is a multiple of 101, hence it might be expected that 809 is a factor of $2^{101} - 1$, but on trial this is found not to be the case, the division yielding a remainder of 491. Fermat's method gives a factor for some values of $2^p - 1$ but it fails in others.

The factorization of large numbers is a very difficult subject. Some values of $2^p - 1$ have large factors; see Bulletin of American Mathematical Society for December, 1903, where Cole shows that 193 707 721 and 761 838 257 287 are the factors of $2^{67} - 1$. For a very interesting history of the work done on Mersenne's numbers see Ball's Mathematical Recreations and Essays.

## 151

In 1850 the Rev. T. P. Kirkman proposed the following problem in the *Lady and Gentlemen's Diary*, an annual published in England: A schoolmistress takes her fifteen girls out for a walk every day in the week; they are arranged in five rows, each row containing three girls; how can they be arranged for a full week so that no girl will walk with any of her schoolmates more than once?

This is generally known as Kirkman's School Girls Problem, and it has been discussed by many mathematicians. The following is Kirkman's solution:

| Sunday | Monday | Tuesday | Wednesday | Thursday | Friday | Saturday |
|---|---|---|---|---|---|---|
| $a_1a_2a_3$ | $a_1b_1c_1$ | $a_1d_1e_1$ | $a_1b_2d_2$ | $a_1c_2e_2$ | $a_1b_3e_3$ | $a_1c_3d_3$ |
| $b_1b_2b_3$ | $a_2b_2c_2$ | $a_2d_2e_2$ | $a_2b_3d_3$ | $a_2c_3e_3$ | $a_2b_1e_1$ | $a_2c_1d_1$ |
| $c_1c_2c_3$ | $a_3d_3e_2$ | $a_3b_3c_3$ | $a_3c_1e_1$ | $a_3b_1d_1$ | $a_3c_2d_3$ | $a_3b_2e_2$ |
| $d_1d_2d_3$ | $b_3d_1e_3$ | $d_3b_1c_2$ | $b_1c_3e_2$ | $c_1b_3d_2$ | $b_2c_3d_1$ | $c_2b_3e_1$ |
| $e_1e_2e_3$ | $c_3d_2e_1$ | $e_3b_2c_1$ | $d_1c_2e_3$ | $e_1b_2d_3$ | $e_2c_1d_3$ | $d_2b_1e_2$ |

He also showed that there are four other solutions, so that the schoolmistress might take out her fifteen young ladies every day for five weeks without any girl walking with any of her mates more than once in a triplet. Ball's Mathematical Recreations and Essays devotes 31 pages to this problem but gives no clear solution of it. In 1862, Sylvester claimed that the girls could walk every day for thirteen weeks under the final condition of the problem.

## 152

### DETERMINANTS

A determinant is an abridged notation for certain algebraic operations to be performed. The theory arose from the formulas required for solving simultaneous equations of the first degree. Thus, when there are two equations containing two unknown quantities,

$$a_1x + b_1y = c_1, \qquad a_2x + b_2y = c_2,$$

the solution gives for the values of $x$ and $y$,

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

which may be written in abridged notation as follows:

$$x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The quantities enclosed within the parallel lines are the simplest possible determinants. $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ means that the quantities in the diagonal inclined downward to the right are to be multiplied together giving $a_1c_2$, and that from this is to be subtracted the product of the quantities in the other diagonal or $a_2c_1$; thus, $a_1c_2 - a_2c_1$ is the value of this determinant.

Again let there be three equations having three unknown quantities, namely,

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3.$$

Here the value of $x$ in the determinant notation is

$$x = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

in which the first determinant has the value

$$b_1 \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} - b_2 \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} + b_3 \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

and this becomes by expanding each minor determinant,

$$b_1 (c_2d_3 - c_3d_2) - b_2 (c_1d_3 - c_3d_1) + b_3 (c_1d_2 - c_2d_1).$$

Similarly, the second determinant in the value of $x$ is

$$a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1).$$

Here the signs of the second, fourth, etc., minor determinants must be minus, while the others are plus.

Determinants have many interesting properties, only one of which can here be mentioned. If any two adjacent rows or adjacent columns be interchanged, the sign of the determinant changes. Thus the three determinants:

$$\begin{vmatrix} +3 & -1 & 0 \\ +4 & +2 & +6 \\ +1 & -3 & +2 \end{vmatrix} \quad \begin{vmatrix} -1 & +3 & 0 \\ +2 & +4 & +6 \\ -3 & +1 & +2 \end{vmatrix} \quad \begin{vmatrix} +3 & +1 & 0 \\ +1 & -3 & +2 \\ +4 & +2 & +6 \end{vmatrix}$$

have the values $+68$, $-68$, and $-68$. When one or more of the quantities in a column is 0, it is convenient to interchange so as to make this the first column, and no change of

sign will result if an even number of columns is passed over, thus:

$$\begin{vmatrix} 0 & +3 & -1 \\ +6 & +4 & +2 \\ +2 & +1 & -3 \end{vmatrix} = 0 - 6(-9+1) + 2(6+4) = +68.$$

## 153

The notation "$a \equiv b \bmod c$" means that $a - b$ is exactly divisible by $c$. Thus, $2^{n-1} \equiv 1 \bmod n$ is a true statement whenever $n$ is a prime number greater than 2; for example, let $n = 7$, then $2^6 - 1 = 63$ which is exactly divisible by 7. The expression $a \equiv b \bmod c$ is called a congruence and the abbreviation "mod" means modulus, it being said that $a$ and $b$ are congruent for the modulus $c$. This notation is mentioned here because many students who see it in mathematical literature know not its meaning.

While it is true that $2^{n-1} \equiv 1 \bmod n$ when $n$ is an odd prime, it is not true that $n$ must be a prime to satisfy the congruence. In fact $2^{340} \equiv 1 \bmod 341$, although 341 is not a prime number. These facts and numerous other interesting ones are proved in that branch of Higher Algebra known as the theory of numbers.

## 154

To find the date of Easter Sunday for any year in the twentieth century: Divide the given year by 19 and call the remainder $a$, divide it by 4 and call the remainder $b$, divide it by 7 and call the remainder $c$; divide $19a + 24$ by 30 and call the remainder $d$; divide $2b + 4c + 6d + 5$ by 7 and call the remainder $e$. Then Easter is March $(22 + d + e)$ or April $(d + e - 9)$. In 1954 and 1981, however, Easter comes one week later than the date given

by this rule. Example: To find Easter for 1916: here $a = 16$, $b = 0$, $c = 5$, $d = 28$, $e = 4$; then the date of Easter Sunday is April 23. The earliest date on which Easter can fall is March 22 and the latest date is April 25.

## 155

### HOW TO DRAW A STRAIGHT LINE

The pantograph used for copying maps on an enlarged or reduced scale is the simplest apparatus in which one point always moves parallel to another. Fig. 58 shows its simplest form, $ABCD$ being a parallelogram with joints at the corners and fastened to the paper at $A$. The straight rod $EFP$ is fixed to $AD$ at $E$ and can slide longitudinally along a peg $F$ which is fixed to $BC$ at $F$. When the point $p$ is moved along a line of the map a pencil at the point $P$ traces a line everywhere parallel to $p$ but on an enlarged scale. When the line followed by $p$ is straight the point $P$ traces a straight parallel line.

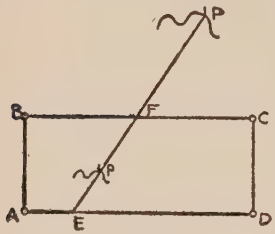


Fig. 58

Fig. 59 shows the principle of an apparatus by which a point $E$ can be made to move in a straight line to $A$. The bar $ED$ slides at $D$ in the fixed groove $DB$ along a straight line which always passes through the fixed point $A$, and the length $AC$ equals $EC$ or $CD$. As the point $C$ is made to rotate about $A$ in a circle $Cc$ the end $D$ slides along the

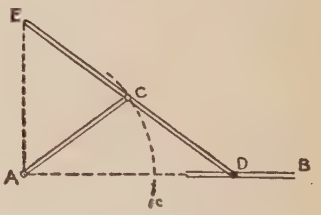


Fig. 59

groove and thus the point $E$ moves in the straight line $EA$.

Euclid's postulate that a straight line can be drawn gives no information as to how this may be done. It is usually supposed to be drawn by means of a ruler, but then the question arises as to how the ruler has been made straight. In 1864 this problem was solved by Peancellier by means of a linkwork in which a point $C$ moves in a circle $Cc$ about a fixed point $B$ while another point $E$ traces a straight line $ee$. Here the parallelogram $CDEF$ is jointed at the corners, and the radius arm $BC$ is one-half of the distance $CA$, and $A$ is also a fixed point which is connected with the parallelogram by the rods $AD$ and $AF$. Thus the circular motion of $C$ causes the right-line motion of $E$. Since there can be no doubt about the circle, this apparatus solves the problem as to how a straight line can be drawn.

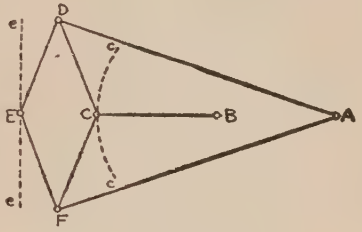


Fig. 60.

## 156

### MISCELLANEOUS FALLACIES

Plot the following points on the rectangular coördinate system and join the successive points by a curve; the first number in each parenthesis being the value of $x$ and the second the value of $y$: (7.0, 4.5), (6.0, 4.7), (4.0, 4.5), (2.0, 3.6), (1.0, 2.9), (0.6, 2.7), (0.3, 2.0), (0.6, 1.0), (1.0, 0.7), (2.0, 0.6), (3.0, 0.8), (3.5, 1.8), (4.0, 0.7), (4.5, 0.6), (5.0, 0.6), (4.8, 1.6), (5.0, 1.8), (5.2, 2.0), (5.5, 2.5), (7.0, 2.6), (7.5, 2.6), (7.7, 2.9), (7.3, 2.9), (7.0, 3.5), (7.0, 4.5). When

the reader has drawn this interesting curve, let it receive careful study.

### 157

The camber of a bridge is a slight upward curve given to the floor so that the structure may have the appearance of strength and stiffness. In the days of early American engineering an excessive camber was given to a certain railroad bridge; it is said that the superintendent received reports that the piers were sinking so as to leave the middle of the structure higher up. Whether this story be true or not, it is certain that the following was clipped from a newspaper printed in 1879 in the Pennsylvania German region along the Delaware River:

"It's warm, Louis, ain't it," said Tod Hartzell to Louis Rapp, as they met on the Delaware bridge yesterday. "Oh, vell, it is," said Louis, "how much you vay now Tod?" "Only 288 pounds," said Tod. "I can beat that," said Louis, "for I vay 294 pounds." The bystanders, by mental arithmetic, added the weights of the men, and then hurried from the bridge which cracked and groaned under the enormous load, while Louis and Tod gracefully moved from the center of the arch which then sprung back to its original position.

### 158

The following fallacy is taken from an old newspaper where it is dignified by the title "A Scientific Lecture on Glass."

A neat, simple, and quick way of punching a hole through a glass plate is, I venture to say, ladies and gentlemen, unknown to most of you. Nothing can be easier than to punch such a hole and at the same time to cause the utter destruction of the glass, but it is not of this that I am to speak, but rather of a simple scientific operation by which anyone may punch or drill a small hole through a plate of window glass, without injuring it or cracking it in the slightest degree. The tools necessary for this purpose are two sets of punches, an old file, and a heavy hammer, which every mechanic possesses. Armed with these, each of you may become skilled in this most interesting and useful accomplishment.

The thicker the pane of glass on which you are to operate, the easier is the process. Having selected it, you choose a place not too near the edge, and with the end of the old file scratch two marks upon it crossing each other like the letter *x*. Then turn the plate over and precisely opposite scratch a similar cross. Next select two set punches of the same size and fasten one of them securely in a vise. Let an assistant hold the plate in a horizontal position with the lower cross resting exactly on the fastened vertical punch, while you with the left hand hold the other punch on the upper cross, and with your right hand grasp the heavy hammer. You then elevate the hammer, but when you strike be careful to give only a moderate blow, for a violent one might cause the destruction of the glass. The effect of the blow, if it be scientifically directed, will be to cause a very slight indentation in the glass. Then let your assistant turn the plate over and again balance it upon the fastened punch, while you with the hammer repeat the careful blow. The indentation will now be more marked than before, and by repeating the process half a dozen times a hole will be made entirely through the glass plate which will be as finely cut as if produced by a swiftly moving rifle ball, while no crack will appear.

When I was captured by the Waldamites this accomplishment proved of the greatest benefit to me, and in fact enabled me ultimately to escape. These singular people, although excellent glass workers, knew no way of cutting it, and when they saw how readily I punched such holes, they not only made obeisance before me, but what was better, they gave me boiled rice and roc's eggs to eat, which were very acceptable, as for six weeks I had eaten only roots with now and then an herb.

## QUOTATIONS

### 159

*Ἐν τοῖς ορθογωὸνιοίς τριγωνοις το ἀπο τῆς τὴν ὀθὴν γωνίαν ὑποτεινούσης πλενρᾶς τέτρὰγωνόν ίσον εστί τοῖς ἀπο των τὴν ορθὴν γονιὰν πἔριεχουσὼν πλενρὼν τετραγώνοις.*

EUCLID, Elements, Book I, 300 B. C.

### 160

*Πᾶς κύκλος ισος εστι τριγώνῳ ορθογωνιῳ οὗ ἡ μὲν ἑκ κέντρον ισῃ μιᾷ των περι τὴν δρθήν, ἡ δε περίῃετρος τῇ βασει.*

ARCHIMEDES, Measurement of the Circle, 220 B. C.

Translation by Haurer, 1798: Jeder Kreis ist einem rechtwinklichen Dreyek gleich, dessen eine Seite um den

rechten Winkle dem Halbmesser und die andere dem Umfang des Kreises gleich ist.

161

Sic incertum, ut, stellarum numerus par an impar sit . . . .

CICERO, Academia, about 50 B. C.

162

This is the third time; I hope, good luck lies in odd numbers. . . . They say, there is divinity in odd numbers, either in nativity, chance, or death.

SHAKESPEARE, Merry Wives of Windsor, 1599.

163

And now we might add something concerning a certain most subtile Spirit which pervades and lies hid in all gross bodies, by the force and action of which Spirit the particles of bodies mutually attract one another at near distances and cohere, if contiguous; and electric bodies operate to greater distances, as well expelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this Spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain to the muscles. But these are things that cannot be explained in a few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic Spirit operates.

NEWTON, Principia, Book III, 1687; American edition, 1848.

## 164

Vedete hora quanto mirabilmenti si accordano cól sistema Copernicano queste tre prime corde, che da principio parevan si dissonanti. Di qui potra instanto . . . vedere con quanto probabilità si porsa concludere, che non la terra, ma il Sole sia nel centra delle conversioni de i pianetti. E poichè la terra vien collocata tra i corpi mondani, che indubitatamente si muovono intorno al Sole, cioè sopra Mercurio, e Venere, e sotto a Saturno, Giove, e Marti, comme parimente non sara probabilissimo, e forse necessario concedera, che essa ancora gli vadia interno?

GALILEI, Third Dialogue, 1630.

## 165

In philosophia experimentali, propositiones ex phaenonenis per inductionem collectae, non obstantibus contrariis hypothesibus, pro veris aut accurate aut quamproxime haberi debent, donec alia occurrerint phaenomena per quas aut accuratiores reddantur aut exceptionibus obnoxiae.

NEWTON, Principia, Book III, 1687.

## 166

It is said that the Egyptians, Persians, and Lacedæmonians seldom elected any new kings but such as had some knowledge in the mathematics; imagining those who had not, to be men of imperfect judgements, and unfit to rule and govern.

Though Plato's censure that those who did not understand the 117th proposition of the 13th book of Euclid's Elements ought not to be ranked among rational creatures, was unreasonable and unjust, yet to give a man character

of universal learning, who is destitute of a competent knowledge in the mathematics, is no less so.

FRANKLIN, Usefulness of Mathematics, 1735.

### 167

Dieu parle, et le chaos se dissipe à sa voix:
Vers un centre commun tout gavite à la fois.
Ce ressort si puissant, l'ame de la nature,
Etait enseveli dans une nuit obscure:
Le compas de Newton, mesurant l'univers,
Leve enfin ce grand voile, et les cieux sout ouverts.

VOLTAIRE, Letter to Madame Châtelet, 1735.

### 168

On s'imagine que toutes ces etoiles, prises ensemble, ne constituent qu'une tres-petite partie se l'univers tout entirer a l'egard duque ces terribles distances ne sout par plus grandes qu'un grain de sable par rapport a la terre. Toute cette immensible est l'ouvrage du Tout-Puissant, qui governe e'galement les plus grandes corps, comme les plus petits, et qui dirige le succès des armes, auquel nous sommes intéréssés.

EULER, Letters to a German Princess, 1760.

### 169

Geheimnissvoll am lichten Tag
Lässt sich Natur des Schleiers nicht berauben,
Und was sie deinem Geist nicht offenbaren mag,
Das zwingst du ihr nicht ab mit Hebeln und mit Schrauben.

GOETHE, Faust, Part I, 1790.

### 170

Der Gebrauch einer unendlichen Grösse als eine Vollendeten ist in der Mathematik niemals erlaubt. Das Unendliche ist nur eine Façon de parler, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kom-

nen als man will, während anderen ohne Einschränkung zu wachsen verstattet ist.

GAUSS, Letter to Schumacher, 1831.

### 171

Then Rory, the rogue, stole his arm round her neck,
So soft and so white, without freckle or speck;
And he look'd in her eyes, that were beaming with light,
And he kissed her sweet lips — don't you think he was right?
"Now, Rory leave off, sir, you'll hug me no more,
That's eight times today that you've kissed me before."
"Then here goes another," says he, "to make sure,
For there's luck in odd numbers," says Rory O'More.

LOVER, Rory O'More, 1839.

### 172

There are terms which cannot be defined, such as number and quantity. Any attempt at a definition would only throw a difficulty in the student's way, which is already done in geometry by the attempts at an explanation of the terms point, straight line, and others, which are to be found in treatises on that subject. A point is defined to be that "which has no parts and which has no magnitude"; a straight line is that which "lies evenly between its extreme points." . . . In this case the explanation is a great deal harder than the term to be explained, which must always happen whenever we are guilty of the absurdity of attempting to make the simplest ideas yet more simple.

DE MORGAN, On the Study of Mathematics, 1831.

### 173

All the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim

of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers.

MAXWELL, Faraday's Lines of Force, 1853.

**174**

Pour les astres en général at pour les grand Comètes en particulier, trois mille ans ne sont pas grand'chose: dans le calendrier de l'éternité c'est moins qu' une seconde. Mais pour l'homme vous savez comme moi, mathematicien lecteurer, que trois mille ans c'est boucoup, beaucoup!

FLAMMARION, Recits de l'infini, 1892.

**175**

There still remain three studies suitable for freemen. Calculation in arithmetic is one of them; the measurement of length, surface, and depth is the second; and the third has to do with the revolutions of the stars in reference to one another.

PLATO, Republic, 350 B. C., Jowett's Translation, 1894.

**176**

The heavens themselves, the planets, and this centre,
Observe degree, priority, and place,
Insisture, course, proportion, season, form,
Office, and custom, in all line of order;
And therefore is the glorious planet, Sol,
In noble eminence enthron'd and spher'd
Amidst the others; whose med'cinable eye
Corrects the ill aspects of planets evil,
And posts, like the commandment of a king,
Sans check, to good and bad: but when the planets
In evil mixture to disorder wander,
What plagues and what portents? what mutiny?
What raging of the sea? frights, changes, horrors,
Divert and crack, rend and deracinate
The unity and married calm of states
Quite from their fixture.

SHAKESPEARE, Troilus and Cressida, Act I, Scene 3, 1602.

177

Lassune' ke' nipune' ani tis de machir mirive' iche manir se' de évenir toné chi amiché ze forime' to viche tarviné.

FLOURNOY, Des Indes a la planete Mars, 1900.

178

The following is one of the many stories told of "old Donald McFarlane," the faithful assistant of Sir William Thomson: The father of a new student when bringing him to the university, after calling to see the Professor (Thomson) drew his assistant to one side and besought him to tell him what his son must do that he might stand well with the Professor. "You want your son to stand weel with the Profeesorr?" asked McFarlane. "Yes." "Weel, then he must just have a guid bellyful o' mathematics!"

S. P. THOMPSON, Life of Lord Kelvin, 1910.

179

Todhunter was not a mere mathematical specialist. He was an excellent linguist; besides being a sound Latin and Greek scholar, he was familiar with French, German, Spanish, Italian, and also Russian, Hebrew, and Sanskrit.

MACFARLANE, Ten British Mathematicians of the Nineteenth Century, 1916.

180

The Appendix is the Soul of a Book.

Old Proverb, n. d.

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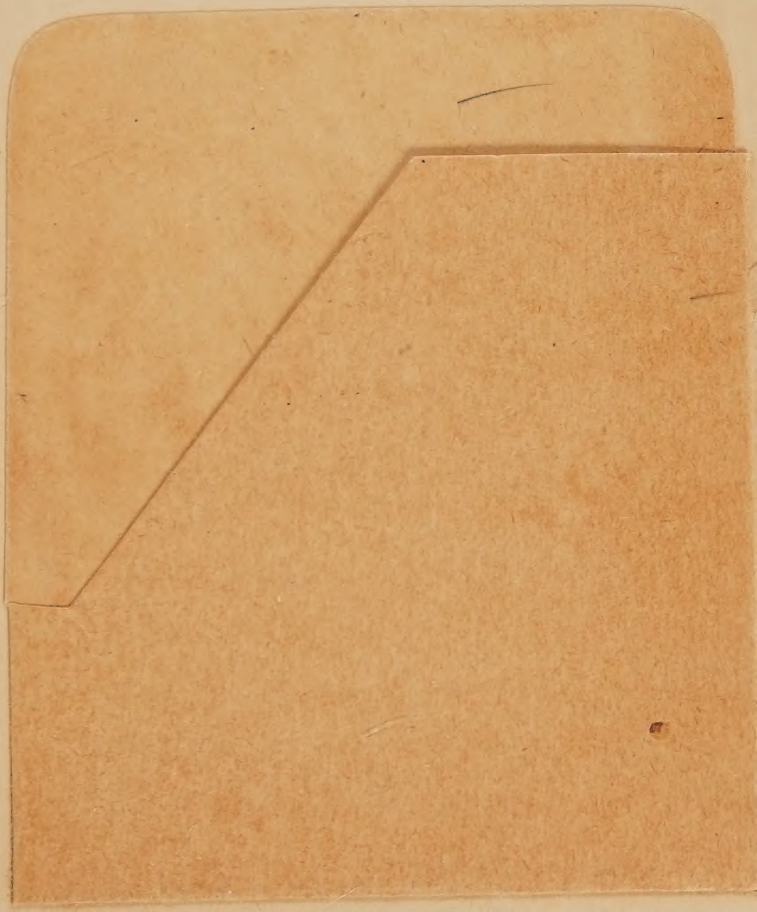

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